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Treatment of the Interface Between Fine Elastic Structures and Fluids with Homogenization Method

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1 Introduction

We study a mechanical system consisting of a fluid and a rapidly oscillating elastic fine structure interacting with the fluid. The goal is to obtain averaged equations which effectively describe the behavior of the system.

This investigation is motivated by modeling a surface acoustic wave sensor based on the generation and detection of horizontally polarized shear waves (see [1]). Acoustic shear waves are excited through an alternate voltage applied to electrodes deposited on a quartz crystal substrate. The waves are transmitted into a thin isotropic guiding layer covered by a thin gold film that contacts a liquid containing a protein to be detected. The protein adheres to a specific receptor, aptamer, immobilized on the surface of the gold film. The arising mass loading causes a phase shift in the electric signal to be measured by an electronic circuit.

One can impress the aptamer-protein layer as a periodic bristle or pin structure on the top of the gold film contacting with the liquid (see Figure 1). The thickness of the aptamer-protein layer is about $4\,nm$, and the number of bristles per surface unit is enormous large. Therefore, the direct numerical modeling of such a structure using fluid-solid interface conditions is impossible. Proper models can be derived using the homogenization technique from [2], [3], [4], and [5] along with the strict treatment of the solid-fluid interface (see e.g. [6]).

2 Mathematical model

The coupled mechanical system under consideration is shown in Figure 1. The solid part consist of a substrate and pins located on its top. The pin structure is assumed to be periodic in the plane (x_1, x_2) and independent of x_3. The domain of the coupled system is denoted by $\Omega \subset \mathbb{R}^3$. For simplicity, we suppose that Ω is the cube $\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, x_k \in (-1;+1),\ k = 1, 2, 3\}$. The domains occupied by the fluid and elastic continua are denoted by Ω_F and Ω_S, respectively; the boundary separating the continua by Γ . Thus, $\Omega = \Omega_F \cup \Gamma \cup \Omega_S$. Let $(\partial\Omega)_F = \partial\Omega \cap \overline{\Omega}_F$ and $(\partial\Omega)_S = \partial\Omega \cap \overline{\Omega}_S$. Then the sets $\Gamma \cup (\partial\Omega)_F$ and $\Gamma \cup (\partial\Omega)_S$ are the boundaries of the domains Ω_F and Ω_S, respectively.

2.1 Governing equations

Assume that the fluid is weakly compressible, which is physically correct because the operation frequency of the coupled structure lies in the acoustic range and the displacements of the fluid

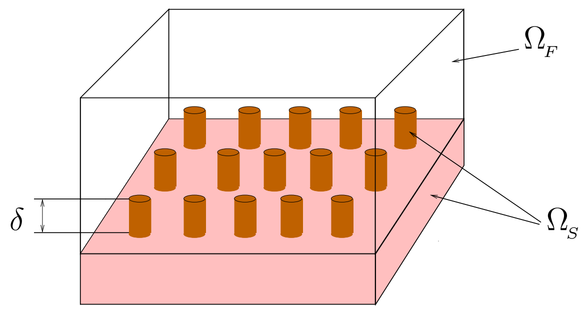

Figure 1: *Coupled system:* $\Omega = \Omega_F \cup \Gamma \cup \Omega_S$.

particles are small. This is a typical acoustic approximation which additionally utilizes linearized Navier-Stokes equations (see [8]). The solid part of the system will be described using the linear elasticity approach. This linear setting is supplemented by the assumption that the domains Ω_F and Ω_S remain unchangeable. Therefore, the coupled mechanical system is described by the following equations

$$\rho_F \boldsymbol{u}_t = -\nabla p + \mathrm{div} P \boldsymbol{u}_{\boldsymbol{x}} + \rho_F \boldsymbol{f} \quad \text{in} \quad \Omega_F, \tag{2.1}$$

$$\gamma p_t = -\mathrm{div} \boldsymbol{u} \quad \text{in} \quad \Omega_F, \tag{2.2}$$

$$\rho_S \boldsymbol{v}_{tt} = \mathrm{div}\, G \boldsymbol{v}_{\boldsymbol{x}} + \rho_S \boldsymbol{f} \quad \text{in} \quad \Omega_S. \tag{2.3}$$

Let $\boldsymbol{n}$ be the normal vector to the fluid-solid interface Γ. No-slip and stress equilibrium conditions on Γ read

$$\boldsymbol{v}_t = \boldsymbol{u} \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$G \boldsymbol{v}_{\boldsymbol{x}} \cdot \boldsymbol{n} = \big(- p\mathcal{I} + P \boldsymbol{u}_{\boldsymbol{x}} \big) \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma \tag{2.5}$$

Boundary and initial conditions are prescribed

$$\boldsymbol{u} = 0 \quad \text{on} \quad (\partial\Omega)_F, \tag{2.6}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad (\partial\Omega)_S, \tag{2.7}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0, \; p|_{t=0} = p^0 \quad \text{in} \quad \Omega_F, \tag{2.8}$$

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}^0, \; \boldsymbol{v}_t|_{t=0} = \boldsymbol{v}'^0 \quad \text{in} \quad \Omega_S. \tag{2.9}$$

Here, ρ_F and ρ_S are the constant densities of the fluid and the solid parts, respectively; $\boldsymbol{u}$ is the velocity field of the fluid, p is the pressure in the fluid, $\boldsymbol{v}$ is the displacement field of the solid part, $\boldsymbol{f}$ is an external force like the gravity. The coefficient γ characterizes the compressibility of the fluid. The fourth-rank tensors $P = \{P_{ijkl}\}$ is defined through the relation

$$P \boldsymbol{u}_{\boldsymbol{x}} = \lambda\, \mathrm{div} \boldsymbol{u}\, \mathcal{I} + \mu\, \mathcal{D}(\boldsymbol{u}). \tag{2.10}$$

The unit tensor I has the components $I_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. The strain velocity tensor $\mathcal{D}(\boldsymbol{u})$ has, as usually, the components $\mathcal{D}_{ij}(\boldsymbol{u}) = 1/2 \left(\partial u_i / \partial x_j + \partial u_j / \partial x_i \right)$. The symbols λ and μ denote positive balk and dynamic viscosity coefficients of the fluid, respectively. As usually, the summation over repeating indices is assumed. The components G_{ijkl} of the elastic stiffness tensor G can be arbitrary up to base restrictions so that arbitrary *anisotropic solids* can be considered.

The no-slip condition (2.4) is the main obstacle for the mathematical treatment of the model (2.1)-(2.9). The method from [9] is used to overcome this difficulty by utilizing the velocity instead of the displacement in equation (2.3). This is being done by introducing the following integral operator

$$\mathcal{J}_t \boldsymbol{w} = \int_0^t \boldsymbol{w}(s)ds$$

that enables to rewrite equation (2.3) in the form

$$\rho_S \boldsymbol{u}_t = \operatorname{div} G \mathcal{J}_t \boldsymbol{u}_{\boldsymbol{x}} + \operatorname{div} \mathcal{G}^0 + \rho_S \boldsymbol{f}, \tag{2.11}$$

where $u = v_t$, $\mathcal{G}^0 = G\boldsymbol{v}^0_{\boldsymbol{x}}$ in Ω_S. Similary, the pressure p can be expressed from equation (2.3) as follows

$$p = -\gamma^{-1}\mathrm{div}\mathcal{J}_t \boldsymbol{u} + p^0 \quad \text{in} \quad \Omega_F. \tag{2.12}$$

Let χ be the characteristic function of the domain Ω_F. Then equations (2.1), (2.2), and (2.3) can be written in the whole domain Ω as one equation with discontinues coefficients

$$\rho\, \boldsymbol{u}_t = \mathrm{div}\boldsymbol{M}^t \boldsymbol{u}_x + \operatorname{div} \mathcal{N}^0 + \rho \boldsymbol{f}, \tag{2.13}$$

where

$$\boldsymbol{M}^t = \chi P + \big(\chi\gamma^{-1} I \otimes I + (1-\chi)G\big)\ \mathcal{J}_t,$$
$$\rho = \rho_F \chi + \rho_S(1-\chi), \quad \mathcal{N}^0 = -\chi p^0 I + (1-\chi)\mathcal{G}^0.$$

The interface condition (2.4) is equivalent to the “continuity” of $\boldsymbol{u}$ on Γ but the condition (2.5) assumes now the form

$$\big(G\mathcal{J}_t \boldsymbol{u}_{\boldsymbol{x}} + \mathcal{G}^0\big) \cdot \boldsymbol{n} = \big(\gamma^{-1}\mathrm{div}\mathcal{J}_t\, \boldsymbol{u} I - p^0 I + P\boldsymbol{u}_{\boldsymbol{x}}\big) \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma, \tag{2.14}$$

accounting (2.12). The boundary and initial data are

$$\boldsymbol{u} = 0 \qquad \text{on } (\partial\Omega)_F, \tag{2.15}$$
$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0 \qquad \text{in } \Omega, \tag{2.16}$$

where the fluid initial condition $\boldsymbol{u}^0$ is extended to Ω_S by letting $\boldsymbol{u}^0(\boldsymbol{x}) = \boldsymbol{v}'^0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega_S$.

A weak formulation of equation (2.13)-(2.16) will be stated.

Remark 2.1 *One can forget the initial distribution $\boldsymbol{v}^0$ of the displacement when considering equation (2.13). It is sufficient to prescribe the initial velocity field $\boldsymbol{u}^0$ in Ω, the initial stress $\mathcal{G}^0$ in Ω_S (this replaces the information about $\boldsymbol{v}^0$), and initial pressure p^0 in Ω_F. The functions $\mathcal{G}^0$ and p^0 yield the function N^0 involved in equation (2.13).*

Remark 2.2 *For mechanical reasons, the tensors P_{ijkl} and G_{ijkl} have the following properties*

$$Z_{ijkl} = Z_{ijlk} = Z_{klij} = Z_{jikl}, \quad Z_{ijkl}\mathcal{V}_{ij}\mathcal{V}_{kl} \geqslant 0,$$

$$Z_{ijkl}\mathcal{V}_{ij}\mathcal{V}_{kl} = 0 \quad \textit{if and only if} \quad \mathcal{V}_{kl} + \mathcal{V}_{lk} = 0 \quad \textit{for all} \quad k,l = 1,2,3.$$

Here, Z_{ijkl} stands for P_{ijkl} or G_{ijkl}.

2.2 Refinement of the original structure

Let us redefine the indicator function χ so that it becomes dependent on a refinement parameter ε Assume that the (x_1, x_2)-projection of the base cell of the pin structure is a square and scale this square to the unit square $\Sigma = [0,1] \times [0,1]$. Denote the scale factor by L. The (x_1, x_2)-projection of the solid part of the base cell will be transformed into a subset $\Sigma_S \subset \Sigma$. Assume that the boundary of Σ_S is smooth, simply connected and does not meet the boundary of Σ. Denote the domain $\Sigma \setminus \overline{\Sigma}_S$ by Σ_F. The domain Σ along with the subdomains Σ_S and Σ_F is called structural cell.

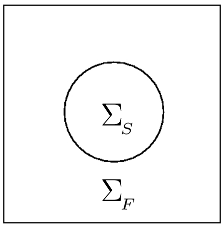

Figure 2: *Structural cell* $\Sigma = [0,1] \times [0,1]$.

Let $\hat{\boldsymbol{x}} = (x_1, x_2)$ and $\hat{\chi}(\hat{\boldsymbol{x}})$ be the Σ-periodic extention of the indicator function of the domain Σ_F to all $\mathbb{R}^2$. We define the modified function χ as follows

$$\chi(\boldsymbol{x}) = \chi(\hat{\boldsymbol{x}}, x_3) = \begin{cases} 1, & x_3 > \delta, \\ \hat{\chi}(\frac{\hat{\boldsymbol{x}}}{\varepsilon}), & -\delta \leqslant x_3 \leqslant \delta, \\ 0, & x_3 < -\delta. \end{cases} \tag{2.17}$$

Remember that δ is the thickness of the pin layer.

Definition 2.3 (Problem S_ε) *Equation (2.13), interface condition (2.14), and initial conditions (2.15) and (2.16) form Problem S_ε. Thereby, the function χ defined by (2.17) is assumed to be involved into the relations (2.13)-(2.16). Thus, the problem is actually dependent on ε.*

Note that the Problem $\mathrm{S}_{\varepsilon_0}$ with $\varepsilon_0 = 1/L$ describes the original coupled structure. If $\varepsilon \to 0$, the pin structure becomes finer and finer laterally, whereas its hight remains constant. The other part of the original structure remains unchangeable.

Definition 2.4 *A function $\boldsymbol{u}$ is called a weak solution to Problem S_ε if*

$$\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega_F)), \quad \mathcal{J}_t\boldsymbol{u} \in L^\infty(0,T; H^1_0(\Omega)) \tag{2.18}$$

and the integral identity

$$\int_0^T \int_\Omega \Big(-\rho\boldsymbol{u}\cdot\boldsymbol{\varphi}_t + \boldsymbol{M}^t\boldsymbol{u}_x : \boldsymbol{\varphi}_x + \mathcal{N}^0 : \boldsymbol{\varphi}_x - \rho\boldsymbol{f}\cdot\boldsymbol{\varphi}\Big) d\boldsymbol{x}dt = \int_\Omega \rho\boldsymbol{u}^0\cdot\boldsymbol{\varphi}^0 d\boldsymbol{x} \tag{2.19}$$

holds for every smooth function $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}|_{t=T} = \boldsymbol{\varphi}|_{\partial\Omega} = 0$.

In this definition and further, T is an arbitrary positive number; the colon denotes the convolution of tensors so that $\mathcal{U} : \mathcal{V} = \mathcal{U}_{ij}\mathcal{V}_{ij}$ for all second-rank tensors $\mathcal{U}$ and $\mathcal{V}$; and the notation f^0 means $f|_{t=0}$. Remark that the second inclusion of (2.18) prevents jumps of $\boldsymbol{u}$ on Γ providing its above mentioned "continuity".

2.3 Solvability of Problem S_ε

It is not difficult to prove existence of a weak solution to Problem S_ε. This question was investigated in [9, sec 9.1], and the following result was established.

Theorem 2.5 *Let $\boldsymbol{u}^0 \in L^2(\Omega)$, $\mathcal{N}^0 \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2([0,T]\times\Omega)$. Then there exists a unique weak solution to Problem S_ε, and the energy estimate holds*

$$\operatorname{ess\,sup}_{t\in(0,T)}\Big(\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)} + \|\mathcal{J}_t\mathcal{D}(\boldsymbol{u})\|^2_{L^2(\Omega_S)}\Big) + \int_0^T \|\mathcal{D}(\boldsymbol{u}(t))\|^2_{L^2(\Omega_F)}\,dt \leqslant C, \tag{2.20}$$

where C is a constant which depends on $\|\boldsymbol{u}^0\|_{L^2(\Omega)}$, $\|\mathcal{N}^0\|_{L^2(\Omega)}$, and $\|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}$ but does not depend on ε.

Due to the Korn inequality we have the following result.

Corollary 2.6 *Under the conditions of Theorem 2.5 there exists an independent of ε constant C such that*

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\mathcal{J}_t\boldsymbol{u}(t)\|_{H^1(\Omega)} \leqslant C. \tag{2.21}$$

Generally speaking, the estimates (2.20) and (2.21) are sufficient to fulfill the homogenization of Problem S_ε due to Proposition 3.3 which will be given below. However, some technical difficulties should be overcome in this case. To avoid that, a stronger estimate for $\boldsymbol{u}$ will be obtained under some compatibility conditions. The next theorem states such a result.

Theorem 2.7 *Let $\boldsymbol{u}^0 \in H^1(\Omega)$, $\mathcal{N}^0 \in L^2(\Omega)$, $\boldsymbol{f}, \boldsymbol{f}_t \in L^2([0,T]\times\Omega)$, and*

$$\operatorname{div}\left(\chi P\boldsymbol{u}^0_{\boldsymbol{x}} + \mathcal{N}^0\right) \in L^2(\Omega). \tag{2.22}$$

Then the weak solution to Problem S_ε satisfies the estimate

$$\operatorname{ess\,sup}_{t\in(0,T)}\big(\|\boldsymbol{u}_t(t)\|_{L^2(\Omega)} + \|\boldsymbol{u}_x(t)\|_{L^2(\Omega)}\big) \leqslant C, \tag{2.23}$$

where C is an independent of ε constant.

Proof. Let us introduce a function $\boldsymbol{w}$ as a solution of the problem

$$\begin{aligned}
&\rho\,\boldsymbol{w}_t = \operatorname{div}\boldsymbol{M}^t\boldsymbol{w}_x + \operatorname{div}\big(\chi\gamma^{-1}I\operatorname{div}\boldsymbol{u}^0 + (1-\chi)G\boldsymbol{u}^0_{\boldsymbol{x}}\big) + \rho\boldsymbol{f}_t,\\
&\rho\,\boldsymbol{w}|_{t=0} = \rho\,\boldsymbol{w}_0 = \operatorname{div}\left(Pu^0_{\boldsymbol{x}} + \mathcal{N}^0\right) + \rho\boldsymbol{f}^0,\\
&\boldsymbol{w}|_{\partial\Omega} = 0.
\end{aligned}$$

The energy estimate for this problem looks as follows

$$\operatorname{ess\,sup}_{t\in(0,T)}\Big(\|\boldsymbol{w}(t)\|^2_{L^2(\Omega)} + \|\mathcal{J}_t\mathcal{D}(\boldsymbol{w})\|^2_{L^2(\Omega_S)}\Big) + \int_0^T \|\mathcal{D}(\boldsymbol{w}(t))\|^2_{L^2(\Omega_F)}\,dt \leqslant C,$$

which gives also

$$\operatorname{ess\,sup}_{t\in[0,T]}\|\mathcal{J}_t\boldsymbol{w}\|_{H^1(\Omega)} < C.$$

The assertion of the theorem is an immediate consequence of these estimates because the function defined as

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_0^t \boldsymbol{w}(\boldsymbol{x},s)ds + \boldsymbol{u}^0(\boldsymbol{x}) = \mathcal{J}_t\boldsymbol{w}(\boldsymbol{x},t) + \boldsymbol{u}^0(\boldsymbol{x})$$

is the solution of Problem S_ε, and $\boldsymbol{u}_t = \boldsymbol{w}$.

According to the definition of $\boldsymbol{u}^0$, the requirement $\boldsymbol{u}^0 \in H^1(\Omega)$ expresses the no-slip condition on Γ at the initial time instant $t = 0$. The requirement (2.22) expresses the stress equilibrium condition on Γ at $t = 0$. From the mechanical point of view, such conditions hold for any time instant including the initial one. Therefore, the requirements of the theorem are feasible.

3 Homogenization of the structure

3.1 Two-scale convergence

Let us denote by $\boldsymbol{u}_\varepsilon$ the solution of Problem S_ε. In order to emphasize the dependence of χ on ε, we denote it by χ^ε. Our goal is to perform the passage to the limit in Problem S_ε as $\varepsilon \to 0$. To do this, we use the two-scale convergence method introduced by G. Nguetseng and developed by other mathematicians (see [11, 2, 3, 10]). Let us formulate main results of this approach adopted to our situation.

Theorem 3.1 *Let $\boldsymbol{w}_\varepsilon$ be a bounded sequence in $L^2\big([0,T] \times \Omega\big)$. There exists a subsequence, still denoted by $\boldsymbol{w}_\varepsilon$, and a function $\overline{\boldsymbol{w}}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}}) \in L^2\big([0,T] \times \Omega \times \Sigma\big)$ such that*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \boldsymbol{w}_\varepsilon(t, \boldsymbol{x}) \boldsymbol{\phi}\big(t, \boldsymbol{x}, \frac{\hat{\boldsymbol{x}}}{\varepsilon}\big)\, d\boldsymbol{x} = \int_0^T \int_\Omega \int_\Sigma \overline{\boldsymbol{w}}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}}) \boldsymbol{\phi}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})\, d\hat{\boldsymbol{\xi}} d\boldsymbol{x} dt$$

for every smooth function $\boldsymbol{\phi}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})$ which is Σ–periodic in $\hat{\boldsymbol{\xi}}$. Such a sequence $\boldsymbol{w}_\varepsilon$ is said to be two-scale converge to $\overline{\boldsymbol{w}}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})$.

Recall the notation $\hat{\boldsymbol{x}} = (x_1, x_2)$ and $\hat{\boldsymbol{\xi}} = (\xi_1, \xi_2)$.

Theorem 3.2 *Let a sequence $\boldsymbol{w}_\varepsilon$ converges weakly to a limit $\boldsymbol{w}$ in $L^2\big(0, T; H^1(\Omega)\big)$. Then $\boldsymbol{w}_\varepsilon$ two–scale converges to $\boldsymbol{w}$ and there exists a function $\overline{\boldsymbol{w}}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})$ in $L^2\big([0,T] \times \Omega; H^1_\#(\Sigma)/\mathbb{R}\big)$ such that $\nabla \boldsymbol{w}_\varepsilon$ two-scale converges to $\nabla_x \boldsymbol{w}(t, \boldsymbol{x}) + \nabla_\xi \overline{\boldsymbol{w}}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})$ up to a subsequence.*

Here $H^1_\#(\Sigma)$ is the space of Σ–periodic functions which belong to the space $H^1(\Sigma)$. Since all functions under consideration do not depend on ξ_3, the notation $\nabla_\xi = (\partial_{\xi_1}, \partial_{\xi_2}, 0)^\top$ is used below.

As a simple application of the theorems stated above, we formulate (without proof) the following result concerning the convergence of solutions of Problem S_ε.

Proposition 3.3 *Let $\boldsymbol{u}_\varepsilon$ be the sequence of solutions to Problem S_ε. Then there exist a subsequence (still denoted by $\boldsymbol{u}_\varepsilon$) and a function $\boldsymbol{u}(t, \boldsymbol{x})$ such that*
1. $\boldsymbol{u}_\varepsilon$ two–scale converges to $\boldsymbol{u}$, and $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in $L^2\big([0,T] \times \Omega\big)$;
2. $\mathcal{J}_t \boldsymbol{u}_\varepsilon$ two–scale converges to $\mathcal{J}_t \boldsymbol{u}$, and $\mathcal{J}_t \boldsymbol{u}_\varepsilon \to \mathcal{J}_t \boldsymbol{u}$ in $L^2\big([0,T] \times \Omega\big)$;
3. $\nabla \mathcal{J}_t \boldsymbol{u}_\varepsilon$ two–scale converges to $\nabla_x \mathcal{J}_t \boldsymbol{u} + \nabla_\xi \boldsymbol{\zeta}$, where $\boldsymbol{\zeta}(t, \boldsymbol{x}, \hat{\boldsymbol{\xi}})$ is a function from $L^2\big([0,T] \times \Omega; H^1_\#(\Sigma)/\mathbb{R}\big)$.

3.2 Passage to the limit in Problem S_ε

Let the initial data of Problem S_ε satisfy the conditions of Theorem 2.7. A solution $\boldsymbol{u}_\varepsilon$ of Problem S_ε satisfies the following integral identity

$$\int_0^T\int_\Omega \Big(- \rho^\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi}_t + \boldsymbol{M}^{\varepsilon t}\boldsymbol{u}_{\varepsilon x} : \boldsymbol{\varphi}_x + \mathcal{N}^{\varepsilon 0} : \boldsymbol{\varphi}_x - \rho^\varepsilon \boldsymbol{f}\cdot\boldsymbol{\varphi}\Big) d\boldsymbol{x}dt = \int_\Omega \rho^\varepsilon \boldsymbol{u}^0 \cdot \boldsymbol{\varphi}^0 d\boldsymbol{x}, \quad (3.1)$$

where ρ^ε, $\boldsymbol{M}^{\varepsilon t}$, and $\mathcal{N}^{\varepsilon 0}$ are defined as in (2.13) but with χ replaced by χ^ε. Let us take

$$\boldsymbol{\varphi}(t,\boldsymbol{x}) = \boldsymbol{\phi}(t,\boldsymbol{x}) + \varepsilon\,\overline{\boldsymbol{\phi}}\Big(t,\boldsymbol{x},\frac{\hat{\boldsymbol{x}}}{\varepsilon}\Big),$$

where $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ are arbitrary functions that vanish for $\boldsymbol{x}\in\partial\Omega$ and at $t=T$. Theorem 3.2 enables the passage to the limit in (3.1) as $\varepsilon\to 0$. The limiting equations look as follows:

$$\int_0^T\int_\Omega\int_\Sigma \Big(- \rho\boldsymbol{u}\cdot\boldsymbol{\phi}_t + \boldsymbol{M}^t(\boldsymbol{u}_x + \overline{\boldsymbol{u}}_\xi) : \boldsymbol{\phi}_x + \mathcal{N}^0 : \boldsymbol{\phi}_x - \rho\boldsymbol{f}\cdot\boldsymbol{\phi}\Big)\, d\hat{\boldsymbol{\xi}}\, d\boldsymbol{x}dt =$$

$$= \int_\Omega\int_\Sigma \rho\boldsymbol{u}^0\cdot\boldsymbol{\phi}^0\, d\hat{\boldsymbol{\xi}}\, d\boldsymbol{x}, \quad (3.2)$$

$$\int_\Sigma \Big(\boldsymbol{M}^t(\boldsymbol{u}_x + \overline{\boldsymbol{u}}_\xi) : \overline{\boldsymbol{\phi}}_\xi + \mathcal{N}^0 : \overline{\boldsymbol{\phi}}_\xi\Big)\, d\hat{\boldsymbol{\xi}} = 0 \quad \text{in} \quad L^2\big([0,T]\times\Omega\big). \quad (3.3)$$

These equations hold for all functions $\boldsymbol{\phi}\in H^1([0,T]\times\Omega)$ and $\overline{\boldsymbol{\phi}}\in H^1_\#(\Sigma)$ such that $\boldsymbol{\phi}$ vanish on $\partial\Omega$ and at $t=T$. The coefficients ρ, $\boldsymbol{M}^t$, and $\mathcal{N}^0$ are defined as in (2.13) with $\chi(\boldsymbol{x})$ replaced by $\chi(\boldsymbol{x},\hat{\boldsymbol{\xi}})$. The function $\chi(\boldsymbol{x},\hat{\boldsymbol{\xi}})$ is defined as in Subsection 2.2:

$$\chi(\boldsymbol{x},\hat{\boldsymbol{\xi}}) = \begin{cases} 1, & x_3 > \delta, \\ \hat{\chi}(\hat{\boldsymbol{\xi}}), & -\delta \leqslant x_3 \leqslant \delta, \\ 0, & x_3 < -\delta. \end{cases}$$

Equation (3.3) is called *cell equation*.

Equations (3.2) and (3.3) are coupled through the auxiliary function $\overline{\boldsymbol{u}}$. The next step consists in finding $\overline{\boldsymbol{u}}$ from the cell equation (3.3) and substituting the obtained expression into equation (3.2).

4 Explicit solving the cell equation

4.1 Operator form of the cell equation in a Hilbert space

It is appropriate to rewrite (3.3) as an equation in the Hilbert space $H = H^1_\#(\Sigma)/\mathbb{R}$ with the inner product

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle = \int_\Sigma \frac{\partial u_i}{\partial\xi_j}\frac{\partial v_i}{\partial\xi_j}\, d\hat{\boldsymbol{\xi}}.$$

The norm in H is denoted by $\|\cdot\|$. Let the operators $\mathcal{A}$ and $\mathcal{B}$ be defined as follows

$$\langle \mathcal{A}\boldsymbol{u},\boldsymbol{v}\rangle = \int\limits_{\Sigma} \chi P_{ijkl}\frac{\partial u_k}{\partial \xi_l}\frac{\partial v_i}{\partial \xi_j}\,d\hat{\boldsymbol{\xi}}, \qquad \langle \mathcal{B}\boldsymbol{u},\boldsymbol{v}\rangle = \int\limits_{\Sigma} \Big(\chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)\,G_{ijkl}\Big)\frac{\partial u_k}{\partial \xi_l}\frac{\partial v_i}{\partial \xi_j}\,d\hat{\boldsymbol{\xi}}$$

for all functions $\boldsymbol{u},\boldsymbol{v}\in H$. Due to the Riesz representation theorem, there exist $\boldsymbol{n}_0$, $\boldsymbol{a}_{kl}$ and $\boldsymbol{b}_{kl}$, $k,l=1,2,3$, such that

$$\langle \boldsymbol{n}_0,\boldsymbol{v}\rangle = \int\limits_{\Sigma} \mathcal{N}^0 : \boldsymbol{v}_\xi\,d\hat{\boldsymbol{\xi}}, \qquad \langle \boldsymbol{a}_{kl},\boldsymbol{v}\rangle = \int\limits_{\Sigma} \chi P_{ijkl}\frac{\partial v_i}{\partial \xi_j}\,d\hat{\boldsymbol{\xi}},$$

$$\langle \boldsymbol{b}_{kl},\boldsymbol{v}\rangle = \int\limits_{\Sigma} \Big(\chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)\,G_{ijkl}\Big)\frac{\partial v_i}{\partial \xi_j}\,d\hat{\boldsymbol{\xi}}$$

for all $\boldsymbol{v}\in H$. Let

$$\boldsymbol{g} = -\big(\boldsymbol{a}_{kl} + \boldsymbol{b}_{kl}\mathcal{J}_t\big)\,\frac{\partial u_k}{\partial x_l} - \boldsymbol{n}_0.$$

Remark that $\mathcal{A}$, $\mathcal{B}$, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{n}_0$ depend on the variable $\boldsymbol{x}$ parametrically just in the same way as the function χ does that, but the dependence on t is absent. The function $\boldsymbol{u}$ can depend on t and x fully arbitrary. Now, the problem (3.3) transforms to the following equation in the space H

$$\mathcal{A}\overline{\boldsymbol{u}} + \mathcal{B}\mathcal{J}_t\overline{\boldsymbol{u}} = \boldsymbol{g}. \tag{4.1}$$

Since the operators $\mathcal{A}$ and $\mathcal{B}$ are trivial whenever $x_3\notin[-\delta,\delta]$, we consider (4.1) for $x_3\in[-\delta,\delta]$, which corresponds to the treatment of the pin layer. In this case, the operators $\mathcal{A}$ and $\mathcal{B}$ are degenerated, therefore, some difficulties appear when solving equation (4.1).

The next section is devoted to the study of the data of (4.1) to prepare tools for its explicit solving.

4.2 Properties of $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{g}$

Proposition 4.1 *The operator $\mathcal{A}$ has the following properties:*
1. $\mathcal{A}$ is a bounded self-adjoint operator on H;
2. $\langle \mathcal{A}\boldsymbol{u},\boldsymbol{u}\rangle \geqslant 0$ for all $\boldsymbol{u}\in H$;
3. The null-space $N(\mathcal{A}) = \{\boldsymbol{u}\in H :\ \boldsymbol{u}$ is constant in $\Sigma_F\}$, and $N(\mathcal{A})^\perp \subset \{\boldsymbol{u}\in H :\ \Delta\boldsymbol{u} = 0$ in $\Sigma_S\}$;
4. There exist positive constants c and C such that

$$c\,\|\boldsymbol{u}\|^2 \leqslant \langle \mathcal{A}\boldsymbol{u},\boldsymbol{u}\rangle \leqslant C\,\|\boldsymbol{u}\|^2 \tag{4.2}$$

for all $\boldsymbol{u}\in N(\mathcal{A})^\perp$;
5. The range $R(\mathcal{A})$ is closed in H, $R(\mathcal{A}) = N(\mathcal{A})^\perp$, and $\mathcal{A}^{-1}$ is defined and bounded as an operator on $R(\mathcal{A})$.

Proof. Assertions 1 and 2 are obvious (see Remark 2.2). The third assertion consists of two parts. In order to prove the first one we have only to establish that

$$N(\mathcal{A}) \subset \{\boldsymbol{u}\in H :\ \boldsymbol{u} \text{ is constant on } \Sigma_F\}$$

because the opposite inclusion is clearly true. Due to the positiveness of the operator $\mathcal{A}$, its null–space consists of function $\boldsymbol{u}$ which satisfy the condition $\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle = 0$. Thus, $\boldsymbol{u} \in N(\mathcal{A})$ implies

$$\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle = \int\limits_{\Sigma} \chi P_{ijkl} \frac{\partial u_k}{\partial \xi_l} \frac{\partial u_i}{\partial \xi_j}\, d\hat{\boldsymbol{\xi}} = 0.$$

Consequently, $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ_{F}, and, hence, $\boldsymbol{u}$ is constant in Σ_{F} because of its periodicity.

Let $\boldsymbol{u} \in N(\mathcal{A})^{\perp}$. By definition, this means that

$$\int\limits_{\Sigma} \frac{\partial u_k}{\partial \xi_l} \frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = \int\limits_{\Sigma_S} \frac{\partial u_k}{\partial \xi_l} \frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = 0$$

for any function $\boldsymbol{v} \in C^{\infty}(\Sigma)$ such that $\boldsymbol{v}$ is constant on $\overline{\Sigma}_{F}$. Consequently, $\boldsymbol{u}$ is harmonic in Σ_{S}, which proves the third assertion.

To validate assertion 3, we need only to prove the left inequality since the right one is obvious. Due to the Korn inequality (see e.g. [12]), there exists a positive constant c_1 such that

$$\int\limits_{\Sigma_F} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}} \leqslant c_1 \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle$$

for every $\boldsymbol{u} \in H$. If $\boldsymbol{u} \in N(\mathcal{A})^{\perp}$, then $\boldsymbol{u}$ is harmonic in Σ_{S}, and there exist positive constants c_2 and c_3 such that

$$c_2 \int\limits_{\Sigma_S} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}} \leqslant \|\boldsymbol{u}\|_{H^{1/2}(\partial\Sigma_S)} \leqslant c_3 \int\limits_{\Sigma_F} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}}.$$

That is, $\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle \geqslant c\,\|\boldsymbol{u}\|^2$ for some constant c.

When proving assertion 5, denote by $\mathcal{A}_{R}$ the restriction of $\mathcal{A}$ to $N(\mathcal{A})^{\perp}$. Due to the estimate (4.2), $R(\mathcal{A}_{R})$ is closed in H. Since $R(\mathcal{A}) = R(\mathcal{A}_{R})$, we conclude that $R(\mathcal{A})$ is also a closed subspace of H. This implies that $N(\mathcal{A})^{\perp} = \overline{R(\mathcal{A})} = R(\mathcal{A})$, and (4.2) is true for $\boldsymbol{u} \in R(\mathcal{A})$. Thus, $\mathcal{A}^{-1}$ exists and is bounded, if $\mathcal{A}$ is considered being restricted to $R(\mathcal{A})$. The proposition is proved.

Proposition 4.2 *The operator $\mathcal{B}$ has the following properties:*
1. $\mathcal{B}$ is a bounded self-adjoint operator on H;
2. $\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle \geqslant 0$ for all $\boldsymbol{u} \in H$;
3. The null-space $N(\mathcal{B}) = \{\boldsymbol{u} \in H :\ \mathcal{D}(\boldsymbol{u}) = 0 \text{ in } \Sigma_{S} \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Sigma_{F}\}$, and $N(\mathcal{B})^{\perp} \subset \{\boldsymbol{u} \in H :\ \Delta \boldsymbol{u} = \nabla q \text{ in } \Sigma_{F} \text{ for some } q \in L^2(\Sigma)\}$;
4. There exist positive constants c and C such that

$$c\,\|\boldsymbol{u}\|^2 \leqslant \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle \leqslant C\,\|\boldsymbol{u}\|^2 \tag{4.3}$$

for all $\boldsymbol{u} \in N(\mathcal{B})^{\perp}$;
5. the range $R(\mathcal{B})$ is closed in H, $R(\mathcal{B}) = N(\mathcal{B})^{\perp}$ and $\mathcal{B}^{-1}$ is defined and bounded as an operator on $R(\mathcal{B})$.

Proof. The first two assertions are obvious. To prove the third one, note that

$$\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle = \int\limits_{\Sigma} \Big(\chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)G_{ijkl}\Big) \frac{\partial u_i}{\partial \xi_j}\frac{\partial u_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = \gamma^{-1}\int\limits_{\Sigma_F} (\operatorname{div}\boldsymbol{u})^2\, d\hat{\boldsymbol{\xi}} + \int\limits_{\Sigma_S} G_{ijkl}\frac{\partial u_i}{\partial \xi_j}\frac{\partial u_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}}$$

for every $\boldsymbol{u} \in H$. Therefore, $\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle = 0$ if and only if $\operatorname{div}\boldsymbol{u} = 0$ in Σ_F and $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ_S.

If $\boldsymbol{u} \in N(\mathcal{B})^{\perp}$, then the equalities

$$0 = \langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int\limits_{\Sigma} \boldsymbol{u}_{\xi}\boldsymbol{v}_{\xi}\, d\hat{\boldsymbol{\xi}} = \int\limits_{\Sigma} \mathcal{D}(\boldsymbol{u}) : \mathcal{D}(\boldsymbol{v})\, d\hat{\boldsymbol{\xi}} = \int\limits_{\Sigma_F} \mathcal{D}(\boldsymbol{u}) : \mathcal{D}(\boldsymbol{v})\, d\hat{\boldsymbol{\xi}}. \tag{4.4}$$

hold for every $\boldsymbol{v} \in N(\mathcal{B})$. Let $\boldsymbol{u}^k \in N(\mathcal{B})^{\perp}$ be a sequence of smooth functions that converges to $\boldsymbol{u}$ in H. Such a sequence exists because $C^{\infty}(\Sigma)$ is dense in $N(\mathcal{B})^{\perp}$. Relation (4.4) is also valid for all $\boldsymbol{u}^k$. If $\boldsymbol{v}$ is an arbitrary smooth function such that $\operatorname{div}\boldsymbol{v} = 0$ and $\operatorname{supp}\boldsymbol{v} \subset \Sigma_F$, then $\boldsymbol{v} \in N(\mathcal{B})$, and

$$0 = \int\limits_{\Sigma_F} \mathcal{D}(\boldsymbol{u}^k) : \mathcal{D}(\boldsymbol{v})\, d\hat{\boldsymbol{\xi}} = -\int\limits_{\Sigma_F} \operatorname{div}\big(\mathcal{D}(\boldsymbol{u}^k)\big)\cdot \boldsymbol{v}\, d\hat{\boldsymbol{\xi}}.$$

Consequently, there exist functions $\tilde{q}^k \in L^2(\Sigma)$ such that $\operatorname{div}\big(\mathcal{D}(\boldsymbol{u}^k) = \nabla\tilde{q}^k$ for all k. Passing to the limit yields $\operatorname{div}\big(\mathcal{D}(\boldsymbol{u}) = \nabla\tilde{q}$. That is $\Delta\boldsymbol{u} = \nabla q$, where $q = \tilde{q} - \operatorname{div}\boldsymbol{u}$. This proves the third assertion.

The right inequality of the fourth assertion is obvious. Let us prove the left one. According to the classical theory of the Stokes equations (see [7, Ch. 4]), the following estimate holds for all $\boldsymbol{u} \in N(\mathcal{B})^{\perp}$:

$$\int\limits_{\Sigma_F} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}} \leqslant c_1\big(\|\operatorname{div}\boldsymbol{u}\|^2_{L_2(\Sigma_F)} + \|\boldsymbol{u}_{\Gamma}\|^2_{H^{1/2}(\partial\Sigma_S)/\mathbb{R}}\big),$$

where $\boldsymbol{u}_{\Gamma}$ is the trace of $\boldsymbol{u}$ on $\partial\Sigma_S$. On the other hand,

$$\|\boldsymbol{u}_{\Gamma}\|^2_{H^{1/2}(\partial\Sigma_S)/\mathbb{R}} \leqslant c_2\int\limits_{\Sigma_S} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}}.$$

Thus, there exists a positive constant c_3 such that

$$\|\boldsymbol{u}\|^2 \leqslant c_3\Big(\int\limits_{\Sigma_S} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}} + \|\operatorname{div}\boldsymbol{u}\|^2_{L_2(\Sigma_F)}\Big) \tag{4.5}$$

for every $\boldsymbol{u} \in N(\mathcal{B})^{\perp}$. In order to obtain (4.3), it is sufficient to prove that there exists a positive constant c_4 such that

$$\int\limits_{\Sigma_S} |\boldsymbol{u}_{\xi}|^2\, d\hat{\boldsymbol{\xi}} \leqslant c_4\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle \tag{4.6}$$

for $\boldsymbol{u} \in N(\mathcal{B})^{\perp}$. This can be done using standard contradiction arguments. Assume the converse, i.e., there exists a sequence $\boldsymbol{u}^n \in N(\mathcal{B})^{\perp}$, $n \in \mathbb{N}$, such that $\int_{\Sigma_S} |\boldsymbol{u}^n_{\xi}|^2\, d\hat{\boldsymbol{\xi}} = 1$ and $\langle \mathcal{B}\boldsymbol{u}^n, \boldsymbol{u}^n\rangle \to 0$ as $n \to 0$. The estimate (4.5) implies that the sequence $\{\boldsymbol{u}^n\}$ is bounded in H too. Thus, there exists its subsequence (still denoted by $\{\boldsymbol{u}^n\}$) that converges weakly in H and $H^1(\Sigma_S)/\mathbb{R}$ but strongly in $L^2(\Sigma)$ to a function $\boldsymbol{u}$. Note that $\boldsymbol{u} \in N(\mathcal{B})^\perp$ since $N(\mathcal{B})^\perp$ is weakly closed in H. Using the Korn inequality yields

$$\int\limits_{\Sigma_S} |\boldsymbol{u}^n_\xi - \boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} \leqslant C \left(\langle \mathcal{B}(\boldsymbol{u}^n - \boldsymbol{u}), \boldsymbol{u}^n - \boldsymbol{u} \rangle + \|\boldsymbol{u}^n - \boldsymbol{u}\|^2_{L^2(\Sigma)} \right).$$

The passage to the limit in this inequality implies that $\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and $\boldsymbol{u}^n \to \boldsymbol{u}$ in H. This means that $\boldsymbol{u} \in N(\mathcal{B})^\perp \cap N(\mathcal{B})$ and $\boldsymbol{u} = 0$ in H. On the other hand, $\int_{\Sigma_S} |\boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} = \lim\limits_{n\to\infty} \int_{\Sigma_S} |\boldsymbol{u}^n_\xi|^2 \, d\hat{\boldsymbol{\xi}} = 1$. This contradiction proves (4.6) and, consequently, (4.3).

The proof of the fifth assertion is the same as for the operator $\mathcal{A}$ in Proposition 4.1.

Proposition 4.3 *The following is true:*

$$\boldsymbol{a}_{kl}, \boldsymbol{b}_{kl}, \boldsymbol{n}_0 \in R(\mathcal{A}) \cap R(\mathcal{B}), \quad k, l = 1, 2, 3.$$

Consequently, $\boldsymbol{g} \in R(\mathcal{A}) \cap R(\mathcal{B})$ for almost all t and $\boldsymbol{x}$, where $\boldsymbol{g}$ is the right-hand-side of the cell equation (4.1).

Proof. Due to Propositions 4.1 and 4.2, $\boldsymbol{w} \in R(\mathcal{A}) \cap R(\mathcal{B})$ iff $\langle \boldsymbol{w}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{v} \in N(\mathcal{A}) \cup N(\mathcal{B})$. Let us verify this condition for $\boldsymbol{a}_{kl}$. The functions $\boldsymbol{b}_{kl}$, and $\boldsymbol{n}_0$ can be treated in the same way. Let $\boldsymbol{v}$ be an arbitrary function from $N(\mathcal{A})$. That is $\boldsymbol{v}$ is a constant in Σ_F because of Proposition 4.1. Thus,

$$\langle \boldsymbol{a}_{kl}, \boldsymbol{v} \rangle = \int\limits_{\Sigma_F} P_{ijkl} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}} = 0.$$

If $\boldsymbol{v} \in N(\mathcal{B})$ then $\mathcal{D}(\boldsymbol{v}) = 0$ in Σ_S according to Proposition 4.2, and

$$\langle \boldsymbol{a}_{kl}, \boldsymbol{v} \rangle = \int\limits_{\Sigma_F} P_{ijkl} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}} = \int\limits_{\Sigma} P_{ijkl} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}} - \int\limits_{\Sigma_S} P_{ijkl} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}} = \int\limits_{\Sigma_S} P_{ijkl} \mathcal{D}_{ij}(\boldsymbol{v}) \, d\hat{\boldsymbol{\xi}} = 0.$$

We used here the periodicity of $\boldsymbol{v}$ in Σ and the symmetry of the tensor P (see Remark 2.1). This proves the proposition.

Proposition 4.4

$$N(\mathcal{A}) \cap N(\mathcal{B}) = \{0\}.$$

Proof. If $\boldsymbol{u} \in N(\mathcal{A}) \cap N(\mathcal{B})$, then $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ due to Propositions 4.1 and 4.2. That is, $\boldsymbol{u}$ is constant in Σ because of its periodicity. This means that $\boldsymbol{u} = 0$ in H.

The result of Proposition 4.4 implies that the operator $\lambda\mathcal{A} + \mathcal{B}$ is invertible for every $\lambda > 0$. Besides that, it is not difficult to see that the operator $(\lambda\mathcal{A} + \mathcal{B})^{-1}$ is bounded in H. Let us introduce the following closed subspaces of H.

$$\begin{aligned} &E_A = (\lambda\mathcal{A} + \mathcal{B})^{-1} R(\mathcal{A}), \\ &E_B = (\lambda\mathcal{A} + \mathcal{B})^{-1} R(\mathcal{B}), \\ &E = E_A \cap E_B = (\lambda\mathcal{A} + \mathcal{B})^{-1} \big(R(\mathcal{A}) \cap R(\mathcal{B}) \big). \end{aligned}$$

Note that the spaces E, $E_{_A}$ and $E_{_B}$ do not depend on λ. More precisely, if $E_{_A}^{\lambda} = (\lambda\mathcal{A} + \mathcal{B})^{-1}R(\mathcal{A})$ then $E_{_A}^{\lambda} = E_{_A}^{\mu}$ for all $\lambda > 0$ and $\mu > 0$. This follows from simple arguments like those. If $\boldsymbol{x} \in E_{_A}^{\lambda}$, then $(\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A})$ and $\mathcal{B}\boldsymbol{x} \in R(\mathcal{A})$. Consequently, $(\mu\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A})$ and $\boldsymbol{x} \in E_{_A}^{\mu}$. That is, $E_{_A}^{\lambda} \subset E_{_A}^{\mu}$. In the same way we can obtain that $E_{_A}^{\mu} \subset E_{_A}^{\lambda}$.

Lemma 4.5 .
The operator $\mathcal{A}$ maps the space $E_{_B}$ into $R(\mathcal{B})$.
The operator $\mathcal{B}$ maps the space $E_{_A}$ into $R(\mathcal{A})$.

Proof. The first part is true due to the following implications: $\boldsymbol{x} \in E_{_B} \Longrightarrow (\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{B}) \Longrightarrow \mathcal{A}\boldsymbol{x} \in R(\mathcal{B})$. The second part is being proved analogously.

Lemma 4.6 *If X is a closed subspace of H then $\mathcal{A}(X)$ and $\mathcal{B}(X)$ are closed in H.*

Proof. Let us verify this assertion for the operator $\mathcal{A}$. Let us take an arbitrary sequence $\boldsymbol{u}_n \in \mathcal{A}(X)$ which converges to a function $\boldsymbol{u}$ in H. There exists a corresponding sequence $\boldsymbol{v}_n \in R(\mathcal{A}) \cap X$ such that $\boldsymbol{u}_n = \mathcal{A}(\boldsymbol{v}_n)$. Due to Proposition 4.1, the operator $\mathcal{A}^{-1}$ is bounded on $R(\mathcal{A})$. This implies that the sequence $\{\boldsymbol{v}_n\}$ converges in H to a function $\boldsymbol{v}$ which is in X because X is closed. In the limit, we have $\boldsymbol{u} = \mathcal{A}(\boldsymbol{v})$. That is, $\boldsymbol{u} \in \mathcal{A}(X)$, which proves the lemma.

Proposition 4.7

$$\mathcal{B}E_{_A} = \mathcal{A}E_{_B} = R(\mathcal{A}) \cap R(\mathcal{B}).$$

That is, for every $\boldsymbol{\psi} \in R(\mathcal{A}) \cap R(\mathcal{B})$, there exist $\boldsymbol{\psi}_{_B} \in E_{_B}$ and $\boldsymbol{\psi}_{_A} \in E_{_A}$ such that $\boldsymbol{\psi} = \mathcal{A}\boldsymbol{\psi}_{_B} = \mathcal{B}\boldsymbol{\psi}_{_A}$.

Proof. Let us prove the first claim. Due to Lemma 4.5, $\mathcal{B}E_{_A} \subset R(\mathcal{A}) \cap R(\mathcal{B})$. Besides that, Lemma 4.6 implies that $\mathcal{B}E_{_A}$ is a closed subspace in H. Suppose that $\mathcal{B}E_{_A} \neq R(\mathcal{A}) \cap R(\mathcal{B})$. Then there exists $\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every $\boldsymbol{y} \in \mathcal{B}E_{_A}$. That is, $\langle \boldsymbol{x}, \mathcal{B}(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{z} \rangle = 0$ for all $\boldsymbol{z} \in H$, and

$$\langle \mathcal{A}(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}\boldsymbol{x}, \boldsymbol{z} \rangle = 0 \quad \text{ for all } \boldsymbol{z} \in H.$$

Consequently, $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}\boldsymbol{x} \in N(\mathcal{A})$ and, hence, $\mathcal{B}\boldsymbol{x} \in (\lambda\mathcal{A} + \mathcal{B})N(\mathcal{A}) = \mathcal{B}N(\mathcal{A})$. That is, there exists $\boldsymbol{y} \in N(\mathcal{A})$ such that $\mathcal{B}\boldsymbol{x} = \mathcal{B}\boldsymbol{y}$ and, therefore, $\mathcal{B}(\boldsymbol{x} - \boldsymbol{y}) = 0$. This implies that $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{y} \in N(\mathcal{B})$. Thus, $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{w}$, where $\boldsymbol{y} \in N(\mathcal{A})$, and $\boldsymbol{w} \in N(\mathcal{B})$. That is, $\boldsymbol{x} \in N(\mathcal{A}) \oplus N(\mathcal{B})$. Consequently, $\boldsymbol{x} = 0$ because $\big(N(\mathcal{A}) \oplus N(\mathcal{B})\big) \cap \big(R(\mathcal{A}) \cap R(\mathcal{B})\big) = \{0\}$. The proposition is proved.

Let us introduce the restrictions $\mathcal{A}_{_E}$ and $\mathcal{B}_{_E}$ of the operators $\mathcal{A}$ and $\mathcal{B}$ to the space E.

Theorem 4.8
1. The operators $\mathcal{A}_{_E}$ and $\mathcal{B}_{_E}$ map E onto $R(\mathcal{A}) \cap R(\mathcal{B})$;
2. The operators $\mathcal{A}_{_E}, \mathcal{B}_{_E} : E \to R(\mathcal{A}) \cap R(\mathcal{B})$ are one-to-one;
3. There exist bounded operators $\mathcal{A}_{_E}^{-1}$, $\mathcal{B}_{_E}^{-1} : R(\mathcal{A}) \cap R(\mathcal{B}) \to E$.

Proof. Let us prove these assertions for the operator $\mathcal{A}_{_E}$ only. The operator $\mathcal{B}_{_E}$ can be treated in the same way.

1. Since $E \subset E_{_B}$, Proposition 4.7 and Lemma 4.6 imply that $\mathcal{A}E \subset R(\mathcal{A}) \cap R(\mathcal{B})$, and $\mathcal{A}E$ is a closed subspace in H. Suppose that $\mathcal{A}E \neq R(\mathcal{A}) \cap R(\mathcal{B})$. This means that there exists $\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every $\boldsymbol{y} \in \mathcal{A}E$. That is,

$$\langle (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \mathcal{A}(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{z} \rangle = 0$$

for all $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$. Thus, due to Proposition 4.7,

$$\langle (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}, \mathcal{B}\boldsymbol{z} \rangle = 0 \quad \text{for all } \boldsymbol{z} \in E_{_A}. \tag{4.7}$$

Since $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x} \in E_{_A}$, we can take $\boldsymbol{z} = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}$. Then the relation (4.7) implies that $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x} \in N(\mathcal{B})$, that is, $\mathcal{A}\boldsymbol{x} \in \mathcal{A}N(\mathcal{B})$. Consequently (see the end of the proof of Proposition 4.7), $\boldsymbol{x} = 0$, which proves the first assertion of the theorem.

2. We have to prove that $N(\mathcal{A}) \cap E = \{0\}$. Let $\boldsymbol{x} \in E$ and $\mathcal{A}\boldsymbol{x} = 0$. Then $\mathcal{B}\boldsymbol{x} = (\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$, that is, $\mathcal{B}\boldsymbol{x} \in R(\mathcal{A})$. But $\boldsymbol{x} \in N(\mathcal{A}) = R(\mathcal{A})^{\perp}$ and, consequently, $\langle \mathcal{B}\boldsymbol{x}, \boldsymbol{x} \rangle = 0$. Since $\mathcal{B}$ is a positive operator, the last relation implies that $\boldsymbol{x} \in N(\mathcal{B})$. Thus, $\boldsymbol{x} \in N(\mathcal{A}) \cap N(\mathcal{B}) = \{0\}$, which proves the second assertion of the theorem.

3. This assertion is the consequence of 1. and 2.. The theorem is proved.

4.3 Solving the cell equation

Now we are in position to find an explicit representation of solutions to the cell equation (4.1). With a new unknown function $\overline{\boldsymbol{\zeta}} = \mathcal{J}_t\overline{\boldsymbol{u}}$, the problem (4.1) assumes the form

$$\mathcal{A}\overline{\boldsymbol{\zeta}}_t + \mathcal{B}\overline{\boldsymbol{\zeta}} = \boldsymbol{g}, \qquad \overline{\boldsymbol{\zeta}}(0) = 0. \tag{4.8}$$

As it follows from Theorem 4.8, the operator $\mathcal{A}_{_E}$ ($\mathcal{A}$ restricted to E) is invertible, the operator $\mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}$ bounded, and $\mathcal{A}_{_E}^{-1}\boldsymbol{g} \in E$. Therefore, the problem

$$\overline{\boldsymbol{\zeta}}_t + \mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}\overline{\boldsymbol{\zeta}} = \mathcal{A}_{_E}^{-1}\boldsymbol{g}, \qquad \overline{\boldsymbol{\zeta}}(0) = 0. \tag{4.9}$$

is uniquely solvable on the subspace E, and the solution is of the form

$$\overline{\boldsymbol{\zeta}}(t) = \int_0^t e^{-(t-s)\mathcal{A}_{_E}^{-1}\mathcal{B}_E}\mathcal{A}_{_E}^{-1}\boldsymbol{g}(s)\,ds. \tag{4.10}$$

Theorem 4.9 *The equations (4.8) and (4.9) are equivalent.*

Proof. Obviously, if $\overline{\boldsymbol{\zeta}}$ is a solution to (4.9), then $\overline{\boldsymbol{\zeta}}$ satisfies (4.8). If $\overline{\boldsymbol{\zeta}}$ is a solution to (4.8), then the function $\overline{\boldsymbol{\eta}} = e^{-\lambda t}\overline{\boldsymbol{\zeta}}$ solves the problem

$$\mathcal{A}\overline{\boldsymbol{\eta}}_t + (\lambda\mathcal{A} + \mathcal{B})\overline{\boldsymbol{\eta}} = e^{-\lambda t}\boldsymbol{g}, \qquad \overline{\boldsymbol{\eta}}(0) = 0. \tag{4.11}$$

Since the operator $\lambda\mathcal{A} + \mathcal{B}$ is non-degenerate for any $\lambda > 0$, we can rewrite (4.11) as follows

$$(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\overline{\boldsymbol{\eta}}_t + \overline{\boldsymbol{\eta}} = e^{-\lambda t}(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{g}, \qquad \overline{\boldsymbol{\eta}}(0) = 0. \tag{4.12}$$

Due to Proposition 4.3, $\boldsymbol{g} \in R(\mathcal{A})$, and, hence $\overline{\boldsymbol{\eta}}(t)$ must belong to $E_{_A}$ for all t. Therefore, $\overline{\boldsymbol{\zeta}}(t) \in E_{_A}$ for all t. On the other hand, equation (4.8) can be rewritten as follows

$$(\lambda\mathcal{A} + \mathcal{B})\overline{\boldsymbol{\zeta}}_t - \mathcal{B}\overline{\boldsymbol{\zeta}}_t + \lambda\mathcal{B}\overline{\boldsymbol{\zeta}} = \lambda\boldsymbol{g}.$$

That is,

$$\overline{\boldsymbol{\zeta}}_t = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}(\overline{\boldsymbol{\zeta}}_t - \lambda\overline{\boldsymbol{\zeta}}) + \lambda(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{g}.$$

Accounting that $\overline{\boldsymbol{\zeta}}(t)$ and $\overline{\boldsymbol{\zeta}}_t(t) \in E_{_A}$ for all t, we establish using Proposition 4.7 that $\overline{\boldsymbol{\zeta}}_t(t) \in E$ for all t. Since $\overline{\boldsymbol{\zeta}}(0) = 0$, we conclude that $\overline{\boldsymbol{\zeta}}(t) \in E$ for all t. Therefore, $\overline{\boldsymbol{\zeta}}$ is a solution of (4.9). The theorem is proved.

Thus, the unique solution of the problem (4.8) is given by formula (4.10). Obviously, a unique solution $\overline{\boldsymbol{u}}$ of the problem (4.1) is given by the formula

$$\overline{\boldsymbol{u}}(t) = \overline{\boldsymbol{\zeta}}_t(t) = \mathcal{A}_{_E}^{-1}\boldsymbol{g}(t) - \mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}\int\limits_0^t e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_{_E}^{-1}\boldsymbol{g}(s)\,ds. \tag{4.13}$$

5 Homogenized structure

5.1 Limiting equations

Substitution of the expression for $\boldsymbol{g}$ into (4.13) gives

$$\overline{\boldsymbol{u}}(t) = -e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_{_E}^{-1}\boldsymbol{n}_0 - \mathcal{A}_{_E}^{-1}\boldsymbol{a}_{kl}\frac{\partial u_k(t)}{\partial x_l} - \int\limits_0^t \boldsymbol{m}_{kl}(t-s)\frac{\partial u_k(s)}{\partial x_l}\,ds, \tag{5.1}$$

$$\mathcal{J}_t\overline{\boldsymbol{u}}(t) = \big(e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E} - \mathcal{I}\big)\mathcal{B}_{_E}^{-1}\boldsymbol{n}_0 - \mathcal{B}_{_E}^{-1}\boldsymbol{b}_{kl}\int\limits_0^t \frac{\partial u_k(s)}{\partial x_l}\,ds - \int\limits_0^t \widetilde{\boldsymbol{m}}_{kl}(t-s)\frac{\partial u_k(s)}{\partial x_l}\,ds, \tag{5.2}$$

where

$$\boldsymbol{m}_{kl}(t) = -\mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\big(\mathcal{A}_{_E}^{-1}\boldsymbol{a}_{kl} - \mathcal{B}_{_E}^{-1}\boldsymbol{b}_{kl}\big) \in E,$$
$$\widetilde{\boldsymbol{m}}_{kl}(t) = e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\big(\mathcal{A}_{_E}^{-1}\boldsymbol{a}_{kl} - \mathcal{B}_{_E}^{-1}\boldsymbol{b}_{kl}\big) \in E.$$

The integration by parts and the formula

$$\frac{d}{ds}e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E} = \mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E}$$

are applied when derivating (5.1) and (5.2). Now we are in position to compute the principal term

$$\int\limits_\Sigma M_{ijkl}^t\frac{\partial\overline{u}_k}{\partial\xi_l}\,d\hat{\boldsymbol{\xi}} = \langle\boldsymbol{a}_{ij},\overline{\boldsymbol{u}}\rangle + \langle\boldsymbol{b}_{ij},\mathcal{J}_t\overline{\boldsymbol{u}}\rangle$$

appearing in the limiting (homogenized) equation (3.2). Utilizing (5.1) and (5.2) and computing other terms of (3.2), we obtain the following limiting equation

$$\int_0^T \int_\Omega \Big(-\rho_\theta u_i \frac{\partial \phi_i}{\partial t} + \big(\theta P_{ijkl} - \alpha_{ijkl}\big) \frac{\partial u_k}{\partial x_l} \frac{\partial \phi_i}{\partial x_j} + \\ + \int_0^t \Big(\theta \gamma^{-1} \delta_{ij}\delta_{kl} + (1-\theta) G_{ijkl} - \beta_{ijkl} + \omega_{ijkl}(t-s)\Big) \frac{\partial u_k}{\partial x_l}\, ds\, \frac{\partial \phi_i}{\partial x_j}\Big)\, d\boldsymbol{x}dt = \\ = \int_0^T \int_\Omega \Big(\rho_\theta f_i \phi_i - \big(\nu_{ij} - \theta p^0 \delta_{ij} + (1-\theta)\mathcal{G}^0_{ij}\big) \frac{\partial \phi_i}{\partial x_j}\Big)\, d\boldsymbol{x}dt + \int_\Omega \rho_\theta \boldsymbol{u}^0 \cdot \boldsymbol{\phi}^0\, d\boldsymbol{x}, \quad (5.3)$$

where

$$\theta(\boldsymbol{x}) = \int_\Sigma \chi\, d\hat{\boldsymbol{\xi}}, \qquad \rho_\theta = \theta \rho_{_F} + (1-\theta)\,\rho_{_S},$$

$$\begin{aligned} \nu_{ij} &= -\langle \boldsymbol{a}_{ij}, e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E} \mathcal{A}_E^{-1} \boldsymbol{n}_0 \rangle + \langle \boldsymbol{b}_{ij}, \big(e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E} - \mathcal{I}\big)\mathcal{B}_E^{-1}\boldsymbol{n}_0\rangle, \\ \alpha_{ijkl} &= \langle \boldsymbol{a}_{ij}, \mathcal{A}_E^{-1} \boldsymbol{a}_{kl}\rangle, \\ \beta_{ijkl} &= \langle \boldsymbol{b}_{ij}, \mathcal{B}_E^{-1} \boldsymbol{b}_{kl}\rangle, \\ \omega_{ijkl}(t) &= -\langle \boldsymbol{a}_{ij}, \boldsymbol{m}_{kl}\rangle - \langle \boldsymbol{b}_{ij}, \widetilde{\boldsymbol{m}}_{kl}\rangle. \end{aligned}$$

Let us denote by $\overline{P}$, $\overline{G}$ and $\mathcal{S}^0$ the tensors with components

$$\overline{P}_{ijkl} = \theta P_{ijkl} - \alpha_{ijkl}, \quad \overline{G}_{ijkl} = \theta\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\theta) G_{ijkl} - \beta_{ijkl},$$

$$\mathcal{S}^0_{ij} = \nu_{ij} - \theta p^0 \delta_{ij} + (1-\theta)\mathcal{G}^0_{ij}.$$

Let us divide the domain Ω into three parts:

$$\Omega^f = \{\boldsymbol{x} \in \Omega \,|\, x_3 > \delta\}, \quad \Omega^s = \{\boldsymbol{x} \in \Omega \,|\, x_3 < -\delta\}, \quad \Omega^h = \{\boldsymbol{x} \in \Omega \,|\, \delta < x_3 < -\delta\}.$$

Let Γ^+_δ be the boundary between Ω^f and Ω^h, Γ^-_δ the boundary between Ω^s and Ω^h. That is $\Omega = \Omega^f \cup \Gamma^+_\delta \cup \Omega^h \cup \Gamma^-_\delta \cup \Omega^s$. Note that $\theta(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \Omega^f$, $\theta(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega^s$, and θ is a constant from the interval $(0,1)$ for $\boldsymbol{x} \in \Omega^h$. As for α_{ijkl}, β_{ijkl}, ν_{ij} and ω_{ijkl}, they are constants for $\boldsymbol{x} \in \Omega^h$ and equal to zero, if $\boldsymbol{x} \in \Omega^f \cup \Omega^s$. So that, the integral identity (5.3) delivers the following equations which should be understand in the distributional sense:

$$\rho_{_F} \boldsymbol{u}_t - \operatorname{div} P \boldsymbol{u}_x - \gamma^{-1} \nabla \operatorname{div} \mathcal{J}_t \boldsymbol{u} = -\nabla p^0 + \rho_{_F} \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega^f, \quad (5.4)$$

$$\rho_{_S} \boldsymbol{u}_t - \operatorname{div} \mathcal{J}_t G \boldsymbol{u}_x = \operatorname{div} \mathcal{G}^0 + \rho_{_S} \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega^s, \quad (5.5)$$

$$\rho_\theta \boldsymbol{u}_t - \operatorname{div} \overline{P} \boldsymbol{u}_x - \operatorname{div} \mathcal{J}_t \overline{G} \boldsymbol{u}_x - \operatorname{div} \int_0^t \omega(t-s) \boldsymbol{u}_x(s)\, ds + \operatorname{div} \mathcal{S}^0 = \rho_\theta \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega^h. \quad (5.6)$$

The natural interfacial boundary conditions at Γ^+_δ and Γ^-_δ can be derived from the integral identity (5.3). Equations (5.4) and (5.5) coincide with (2.1) and (2.11) respectively. That is, the governing equations for the pure fractions do not change after the homogenization, which have been expected. What we have new is an integral-differential equation (5.6) which can not be reduced to a pure differential equation by differentiating or by a substitution like $\boldsymbol{w} = \mathcal{J}_t \boldsymbol{u}$. The operators involved in the equation have to be investigated to confirm the parabolic type of its principal part.

5.2 Investigation of $\overline{P}$ and $\overline{G}$

The main objective of this subsection is to prove the strong positiveness of the tensor $\overline{P}$ and the non-negativeness of $\overline{G}$. The null-space of $\overline{G}$ will be also described.

Proposition 5.1 *For every second-rank tensor $\mathcal{Z}$, the following is valid:*

$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant 0, \qquad \overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant 0.$$

Proof. Let us prove the assertion for $\overline{P}$. Denote $\boldsymbol{z} = \boldsymbol{a}_{ij}\mathcal{Z}_{ij}$. Due to Proposition 4.3, $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$ and, as it follows from Theorem 4.8, there exists a unique $\boldsymbol{y} \in E$ such that $\mathcal{A}_{E}\boldsymbol{y} = \boldsymbol{z}$. This means that

$$\int\limits_{\Sigma} \chi P_{ijkl}\frac{\partial y_i}{\partial \xi_j}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = \langle \boldsymbol{z}, \boldsymbol{v}\rangle = \int\limits_{\Sigma} \chi P_{ijkl}\mathcal{Z}_{ij}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} \tag{5.7}$$

for all $\boldsymbol{v} \in H$. On the other hand, the definition yields

$$\alpha_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = \langle \boldsymbol{a}_{ij}\mathcal{Z}_{ij}, \mathcal{A}_{E}^{-1}\boldsymbol{a}_{kl}\mathcal{Z}_{kl}\rangle = \langle \boldsymbol{z}, \mathcal{A}_{E}^{-1}\boldsymbol{z}\rangle = \langle \mathcal{A}_{E}\boldsymbol{y}, \boldsymbol{y}\rangle.$$

From the last relation and (5.7) with $\boldsymbol{v} = \boldsymbol{y}$, we obtain

$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = \theta\, P_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \mathcal{A}_{E}\boldsymbol{y}, \boldsymbol{y}\rangle = \int\limits_{\Sigma} \Big(\chi P_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \chi P_{ijkl}\frac{\partial y_i}{\partial \xi_j}\frac{\partial y_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}} =$$

$$= \int\limits_{\Sigma} \chi P_{ijkl}\Big(\mathcal{Z}_{ij} - \frac{\partial y_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}}. \tag{5.8}$$

The right-hand side of the last relation is clearly positive, and the required assertion is proved for the tensor $\overline{P}$. Positiveness of the tensor G can be verified in the same way.

The next theorem states the strong positiveness of the tensor $\overline{P}$.

Theorem 5.2 *There exists a positive constant C such that*

$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant C\,|\mathcal{Z}|^2$$

for every second-rank tensor $\mathcal{Z}$. Here $|\mathcal{Z}|^2 = \mathcal{Z}_{ij}\mathcal{Z}_{ij}$.

Proof. Due to the symmetry of $\overline{P}$, it is sufficient to consider symmetric tensors $\mathcal{Z}$ only. Assume that the assertion of the theorem is false. Then there exists a sequence $\{\mathcal{Z}^n\}$ such that $|\mathcal{Z}^n| = 1$ and $\overline{P}_{ijkl}\mathcal{Z}^n_{ij}\mathcal{Z}^n_{kl} \to 0$ as $n \to \infty$. The sequence $\{\mathcal{Z}^n\}$ is compact in $\mathbb{R}^3 \times \mathbb{R}^3$ and, therefore, it has a subsequence denoted again by $\{\mathcal{Z}^n\}$ which converges to a matrix $\mathcal{Z}^0$ such that $|\mathcal{Z}^0| = 1$. This means that the corresponding sequences $\boldsymbol{z}^n$ and $\boldsymbol{y}^n$ defined as $\boldsymbol{z}^n = \boldsymbol{a}_{ij}\mathcal{Z}^n_{ij}$ and $\boldsymbol{y}^n = \mathcal{A}_{E}^{-1}\boldsymbol{z}^n$ converge in H to $\boldsymbol{z}^0$ and $\boldsymbol{y}^0$, respectively. We use here the notations introduced in the proof of the previous proposition. Thus, the relation

$$\overline{P}_{ijkl}\mathcal{Z}^0_{ij}\mathcal{Z}^0_{kl} = \theta\, P_{ijkl}\mathcal{Z}^0_{ij}\mathcal{Z}^0_{kl} - \langle \mathcal{A}_{E}\boldsymbol{y}^0, \boldsymbol{y}^0\rangle = \int\limits_{\Sigma} \chi P_{ijkl}\Big(\mathcal{Z}^0_{ij} - \frac{\partial y^0_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}^0_{kl} - \frac{\partial y^0_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}} = 0.$$

holds due to (5.8). That is,

$$\chi P_{ijkl}\Big(\mathcal{Z}^0_{ij} - \frac{\partial y^0_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}^0_{kl} - \frac{\partial y^0_k}{\partial \xi_l}\Big) = 0 \qquad \text{in} \quad \Sigma,$$

and, consequently, $\mathcal{D}(\boldsymbol{y}^0) = \mathcal{Z}^0$ in Σ_F. This implies that $\mathcal{D}(\boldsymbol{y}^0 - \mathcal{Z}^0\boldsymbol{\xi}) = 0$ in Σ_F. Therefore, $\boldsymbol{y}^0(\boldsymbol{\xi})$ is a linear function of $\boldsymbol{\xi}$ for $\boldsymbol{\xi} \in \Sigma_F$. The only linear function satisfying the periodicity boundary conditions on $\partial\Sigma$ is a constant, which implies that $\mathcal{Z}^0 = 0$. This is impossible because $|\mathcal{Z}^0| = 1$. This contradiction proves the theorem.

Remark that the arguments like those in the proof of Theorem 5.2 do not lead to a contradiction in the case of the tensor $\overline{G}$. The next theorem shows that the tensor $\overline{G}$ is degenerated and describes its null-space.

Theorem 5.3 *The tensor $\overline{G}$ is degenerate and $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ if and only if $\mathcal{Z}_{11} + \mathcal{Z}_{22} = 0$ and $\mathcal{Z}_{33} = 0$.*

Proof. Similarly to the previous theorem, it is sufficient to consider symmetric tensors $\mathcal{Z}$ only. Let us denote $\boldsymbol{z} = \boldsymbol{b}_{ij}\mathcal{Z}_{ij}$. Due to Proposition 4.3 and Theorem 4.8, $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$, and there exist unique elements $\boldsymbol{y}^E \in E$ and $\boldsymbol{y}^R \in R(\mathcal{B})$ such that

$$\mathcal{B}\boldsymbol{y}^E = \boldsymbol{z}, \qquad \mathcal{B}\boldsymbol{y}^R = \boldsymbol{z}. \tag{5.9}$$

It follows that $\boldsymbol{y}^N = \boldsymbol{y}^E - \boldsymbol{y}^R \in N(\mathcal{B})$. Besides that, $\mathcal{B}_E^{-1}\mathcal{B}\boldsymbol{y}^E = \boldsymbol{y}^E$. Therefore,

$$\langle \boldsymbol{z}, \mathcal{B}_E^{-1}\boldsymbol{z}\rangle = \langle \mathcal{B}\boldsymbol{y}^E, \boldsymbol{y}^E\rangle = \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R + \boldsymbol{y}^N\rangle = \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R\rangle.$$

The second equation in (5.9) implies that

$$\int_\Sigma K_{ijkl}(\chi)\frac{\partial y_i^R}{\partial \xi_j}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = \int_\Sigma K_{ijkl}(\chi)\mathcal{Z}_{ij}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}}, \tag{5.10}$$

for all $\boldsymbol{v} \in H$, where

$$K_{ijkl}(\chi) = \chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)G_{ijkl}.$$

As a consequence of this equation, we find

$$\begin{aligned}\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = K_{ijkl}(\theta)\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \boldsymbol{z}, \mathcal{B}_E^{-1}\boldsymbol{z}\rangle = K_{ijkl}(\theta)\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R\rangle = \\ = \int_\Sigma K_{ijkl}(\chi)\Big(\mathcal{Z}_{ij} - \frac{\partial y_i^R}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k^R}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}} = \\ = \int_{\Sigma_F} \gamma^{-1}(\mathrm{tr}\mathcal{Z} - \mathrm{div}\,\boldsymbol{y}^R)^2\, d\hat{\boldsymbol{\xi}} + \int_{\Sigma_S} G_{ijkl}\big(\mathcal{Z}_{ij} - \mathcal{D}_{ij}(\boldsymbol{y}^R)\big)\big(\mathcal{Z}_{kl} - \mathcal{D}_{kl}(\boldsymbol{y}^R)\big)\, d\hat{\boldsymbol{\xi}}.\end{aligned}$$

Remark that (5.10) is the Euler–Lagrange equation for the functional

$$F_z(\boldsymbol{y}) = \int_\Sigma K_{ijkl}(\chi)\Big(\mathcal{Z}_{ij} - \frac{\partial y_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}}.$$

Due to Proposition 4.2 (assertion 4), this functional is strictly convex on $R(\mathcal{B})$, and $\boldsymbol{y}^R$ is its unique minimizer there. That is,

$$\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = F_z(\boldsymbol{y}^R) = \min_{\boldsymbol{y}\in R(\mathcal{B})} F_z(\boldsymbol{y}).$$

Thus, $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ if and only if there exists $\boldsymbol{y}^R \in R(\mathcal{B})$ such that $F_z(\boldsymbol{y}^R) = 0$. It is not difficult to see that $F_z(\boldsymbol{y}) = F_z(\boldsymbol{y} + \boldsymbol{w})$ for every $\boldsymbol{w} \in N(\mathcal{B})$. Since $R(\mathcal{B}) \oplus N(\mathcal{B}) = H$, the existence of $\boldsymbol{y}^R \in R(\mathcal{B})$ with $F_z(\boldsymbol{y}^R) = 0$ is equivalent to the existence of a function $\boldsymbol{y} \in H$ which satisfies the condition $F_z(\boldsymbol{y}) = 0$. Due to the positiveness of the functional F_z, we can conclude that $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ if and only if there exists $\boldsymbol{y} \in H$ such that

$$\operatorname{div} \boldsymbol{y} = \operatorname{tr} \mathcal{Z}, \qquad \text{if} \qquad \hat{\boldsymbol{\xi}} \in \Sigma_F, \tag{5.11}$$

$$\mathcal{D}(\boldsymbol{y}) = \mathcal{Z}, \qquad \text{if} \qquad \hat{\boldsymbol{\xi}} \in \Sigma_S. \tag{5.12}$$

Suppose that the both last conditions are satisfied. Since functions from H do not depend on ξ_3, (5.12) implies that $\mathcal{Z}_{33} = 0$. Moreover, due to (5.12), $\operatorname{div} \boldsymbol{y} = \operatorname{tr} \mathcal{Z}$ in Σ_S. That is $\operatorname{div} \boldsymbol{y} = \operatorname{tr} \mathcal{Z}$ in Σ. Integrating this equality over Σ we find that $\operatorname{tr} \mathcal{Z} = 0$ because $\boldsymbol{y}$ is periodic. Thus, we have proved the assertion of the theorem in one direction (the necessity).

Let us suppose that $\mathcal{Z}_{11} + \mathcal{Z}_{22} = 0$ and $\mathcal{Z}_{33} = 0$. In order to complete the proof of the theorem, we have to prove that there exists a function $\boldsymbol{y} \in H$ satisfying (5.11) and (5.12). Equation (5.12) is easy to solve. Namely, its solution looks as follows

$$\boldsymbol{y}(\boldsymbol{\xi}) = \mathcal{Z}\boldsymbol{\xi} + \mathcal{Q}\boldsymbol{\xi} + \boldsymbol{y}_0, \qquad \hat{\boldsymbol{\xi}} \in \Sigma_S,$$

where $\mathcal{Q}$ is a skew-symmetric matrix, and $\boldsymbol{y}_0$ is a constant which can be dropped because functions from the space H are defined up to a constant. Let us denote $\mathcal{T} = \mathcal{Z} + \mathcal{Q}$. Since functions from H do not depend on ξ_3, we find that $\mathcal{T}_{i3} = 0\,(i = 1, 2, 3)$ and $y_3 = \mathcal{T}_{31}\xi_1 + \mathcal{T}_{32}\xi_2$ for $\hat{\boldsymbol{\xi}} \in \Sigma_S$. We extend y_3 to the whole domain Σ in such a way that it would be a periodic function (assuming equal values on the opposite edges of Σ).

In order to determine y_1 and y_2 in Σ_F, we have to solve the problem

$$\begin{aligned} &\frac{\partial y_1}{\partial \xi_1} + \frac{\partial y_2}{\partial \xi_2} = 0, && \hat{\boldsymbol{\xi}} \in \Sigma_F, \\ &\boldsymbol{y}(\hat{\boldsymbol{\xi}}) = \mathcal{T}\boldsymbol{\xi}, && \hat{\boldsymbol{\xi}} \in \partial\Sigma_S, \\ &y_1 \quad \text{and} \quad y_2 \quad \text{are} \quad \text{periodic} \quad \text{in} \quad \Sigma. \end{aligned}$$

This problem is clearly solvable, and the theorem is completely proved.

As one can see from equation (5.6), the tensor $\overline{G}$ describes elastic stresses in the homogenized continuum. Theorem 5.3 says that the homogenized material has rather strange properties. Namely, it does not resist to the deformation, if the first invariant and the component (3,3) of the corresponding strain tensor are equal to zero. In other words, such deformations do not produce any stresses. The described class of deformations is sufficiently large. It contains all deformations which do not change volume.

6 Dispersion Relations

Dispersion relations express the dependence of the wave velocity on the excitation frequency. Such a dependence is typical when stydying the propagation of acoustic waves in multi-layered anisotropic structures like the biosensor described in the introduction. The computation of dispersion relations is based on the construction of traveling wave solutions that exponentially decrease towards $-x_3$ and $+x_3$ in the substrate and the fluid, respectively. Proper mechanical interface conditions between the layers as well as the interaction with the fluid have to be accounted.

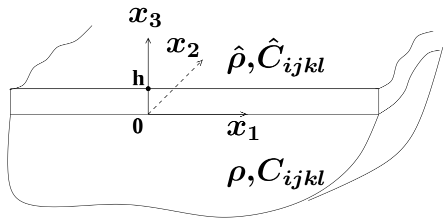

Figure 3: A sample structure. An anisotropic layer lies on an anisotropic half-space substrate. Here, $\hat{\rho}$, ρ, and $\hat{C}_{ijkl}$, C_{ijkl} are the densities and the elastic stiffness tensors, respectively.

For simplicity, consider a simplified structure shown in Figure 3 whose upper layer is free of liquid.

The elasticity equations for the substrate and the upper layer read:

$$\rho v_{i\,tt} - C_{ijkl}\frac{\partial^2 v_l}{\partial x_j \partial x_k} = 0,\ i = 1,2,3, \tag{6.1}$$

$$\hat{\rho}\hat{v}_{i\,tt} - \hat{C}_{ijkl}\frac{\partial^2 \hat{v}_l}{\partial x_j \partial x_k} = 0,\ i = 1,2,3, \tag{6.2}$$

where v_i and $\hat{v}_i$, $i = 1,2,3$, are components of the displacement vectors. A plain wave propagating in the structure in x_1-direction is of the form:

$$v_i(x_1, x_3) = a_i(x_3)\cos(kx_1 - \omega t) + b_i(x_3)\sin(kx_1 - \omega t), \tag{6.3}$$

$$\hat{v}_i(x_1, x_3) = \hat{a}_i(x_3)\cos(kx_1 - \omega t) + \hat{b}_i(x_3)\sin(kx_1 - \omega t). \tag{6.4}$$

Here, k is the wave number, ω the circuit frequency. The substitution of (6.3) and (6.4) into (6.1) and (6.2), respectively, yields

$$-C_{i33l}\,\ddot{a}_l - (C_{i13l} + C_{i31l})\,\dot{b}_l + C_{i11l}\,a_l - \rho\frac{\omega^2}{k^2}\,a_i = 0,$$
$$-C_{i33l}\,\ddot{b}_l + (C_{i13l} + C_{i31l})\,\dot{a}_l + C_{i11l}\,b_l - \rho\frac{\omega^2}{k^2}\,b_i = 0,$$

and

$$-\hat{C}_{i33l}\,\ddot{\hat{a}}_l - (\hat{C}_{i13l} + \hat{C}_{i31l})\,\dot{\hat{b}}_l + \hat{C}_{i11l}\,\hat{a}_l - \hat{\rho}\frac{\omega^2}{k^2}\,\hat{a}_i = 0,$$
$$-\hat{C}_{i33l}\,\ddot{\hat{b}}_l + (\hat{C}_{i13l} + \hat{C}_{i31l})\,\dot{\hat{a}}_l + \hat{C}_{i11l}\,\hat{b}_l - \hat{\rho}\frac{\omega^2}{k^2}\,\hat{b}_i = 0,$$
$$i = 1,2,3.$$

Here, the dot denotes the differentiation with respect to the variable $\tilde{x}_3 = kx_3$. With the state vectors

$$\boldsymbol{p} = (a_1, a_2, a_3, b_1, b_2, b_3, \dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{b}_1, \dot{b}_2, \dot{b}_3)^T \in R^{12},$$

$$\hat{\boldsymbol{p}} = (\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3, \dot{\hat{a}}_1, \dot{\hat{a}}_2, \dot{\hat{a}}_3, \dot{\hat{b}}_1, \dot{\hat{b}}_2, \dot{\hat{b}}_3)^T \in R^{12},$$

the above systems can be rewritten in the normal form as follows:

$$\dot{\boldsymbol{p}} = A\,\boldsymbol{p}, \quad \dot{\hat{\boldsymbol{p}}} = \hat{A}\,\hat{\boldsymbol{p}}, \tag{6.5}$$

where A and $\hat{A}$ are the corresponding matrices. Let $\lambda_1, \lambda_2, ..., \lambda_{12}$ and $\boldsymbol{h}_1, \boldsymbol{h}_2, ..., \boldsymbol{h}_{12}$ (respectively $\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_{12}$ and $\hat{\boldsymbol{h}}_1, \hat{\boldsymbol{h}}_2, ..., \hat{\boldsymbol{h}}_{12}$) be eigenvalues and eigenvectors of A (respectively $\hat{A}$). One can verify that exactly ℓ linear independent eigenvectors can be found for each ℓ-multiple eigenvalue. Therefore, solutions of (6.5) are of the form: $\boldsymbol{p}(x_3) = \sum_{i=1}^{12} D_i \boldsymbol{h}_i e^{\lambda_i k x_3}$, $\hat{\boldsymbol{p}}(x_3) = \sum_{i=1}^{12} \hat{D}_i \hat{\boldsymbol{h}}_i e^{\hat{\lambda}_i k x_3}$, where D_i and $\hat{D}_i$ are arbitrary constants. Selecting decreasing solutions in the substrate yields:

$$\boldsymbol{p}(x_3) = \sum_{j=1}^{N} D_j \boldsymbol{h}_{i_j} e^{\lambda_{i_j} k x_3}, \quad \text{Re}\ \lambda_{i_j} > 0.$$

Note that $N \leq 6$ due to the up-down symmetry of the substrate. Solutions in the upper layer have to be of the oscillatory type:

$$\hat{\boldsymbol{p}}(x_3) = \sum_{j=1}^{L} \hat{D}_j \hat{\boldsymbol{h}}_{i_j} e^{\hat{\lambda}_{i_j} k x_3}, \quad \text{Re}\ \hat{\lambda}_{i_j} = 0.$$

Thus,

$$a_l = \sum_{j=1}^{N} D_j h_{i_j}^{(l)} e^{\lambda_{i_j} k x_3}, \quad b_l = \sum_{j=1}^{N} D_j h_{i_j}^{(l+3)} e^{\lambda_{i_j} k x_3},$$
$$\hat{a}_l = \sum_{j=1}^{L} \hat{D}_j \hat{h}_{i_j}^{(l)} e^{\hat{\lambda}_{i_j} k x_3}, \quad \hat{b}_l = \sum_{j=1}^{L} \hat{D}_j \hat{h}_{i_j}^{(l+3)} e^{\hat{\lambda}_{i_j} k x_3},$$

where l runs from 1 to 3. Therefore, the displacements v_i and $\hat{v}_i$, see (6.3) and (6.4), depend linearly on $D_j, j = 1, N$, and $\hat{D}_l,\ l = 1, L$, respectively.

For all x_1 and t, the following interface conditions must hold:

$$v_i = \hat{v}_i, \quad \text{at } x_3 = 0, \text{ continuity;}$$
$$C_{i3kl}\frac{\partial v_i}{\partial x_k} = \hat{C}_{ijkl}\frac{\partial \hat{v}_i}{\partial x_k}, \quad \text{at } x_3 = 0, \text{ equilibrium of pressures;}$$
$$\hat{C}_{ijkl}\frac{\partial \hat{v}_i}{\partial x_k} = 0, \quad \text{at } x_3 = h, \text{ free of forces boundary.}$$

The last system yields 18 linear equations for $N + L \leq 18$ coefficients D_j and $\hat{D}_l$. Note that $N + L < 18$ as a rule. Let $V = \omega/k$ be the unknown wave velocity and $G(V)$ the $18 \times (N + L)$-matrix of the above system. Feasible wave velocities are determined from the condition of nontrivial solvability for the system $G(V)\boldsymbol{D} = 0$, where $\boldsymbol{D} = (D_1, ...D_N, \hat{D}_1, ...\hat{D}_L)^T$. Thus, the condition $rank\, G(V) < N + L$ holds for the feasible velocities, which is equivalent to the following condition: det $\left|\bar{G}^T(V)G(V)\right| = 0$, $\bar{G}$ being the adjoint matrix. The last equation can be easily solved because the computation of the left-hand-side runs very quickly even on a simple computer. Usually, three roots are being found, which corresponds to three wave types that propagate with

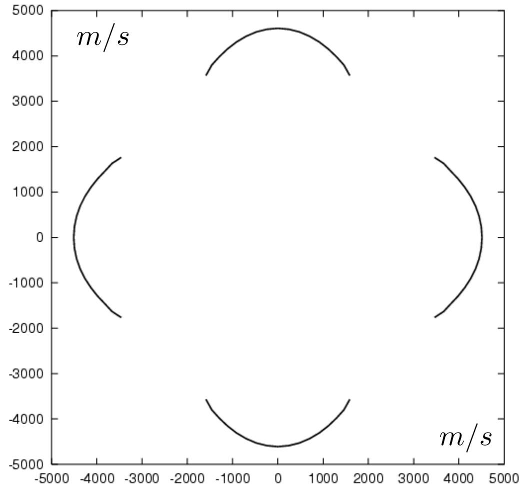

Figure 4: Velocity profile typical for structures with anisotropic substrates possessing rotation symmetries.

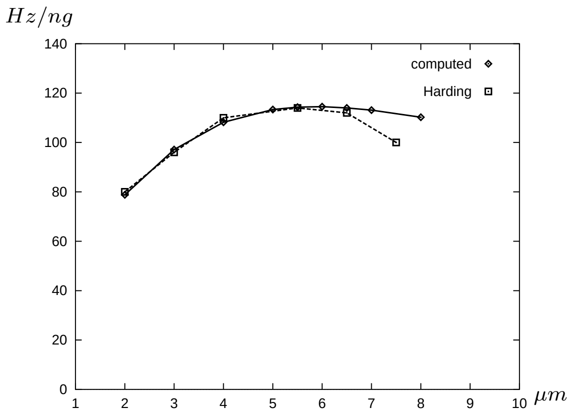

Figure 5: Comparison of numerically computed and experimentally measured sensitivities. Dependence on the thickness of the guiding (upper) layer is shown.

different velocities. The selection of the desired wave type is quite obvious because the relation between their velocities is known.

Figure 4 shows the velocity profile for Love shear waves in the structure shown in Figure 3. The substrate is an ST-cut of α-quartz. The upper layer consists of amorphous quartz. The blank parts of the curve correspond to the absence of Love shear waves for these directions.

Figure 5 represents a verification of the dispersion relations method. The sensitivity of a Love wave sensor based on the structure shown in Figure 3 is being computed as $(\tilde{\omega}_r - \omega_r)/\Delta m$, where $\tilde{\omega}_r$ and ω_r are resonance frequencies for the loaded and unloaded sensor, respectively. The resonance frequency is defined by the equation $\lambda(\omega_r) = 40\mu m.$ Here $\lambda(\omega) = 2\pi V(\omega)/\omega$ is the wave length, and $40\mu m$ is the period of the input electrodes. The loading is modeled through the adding of a $0.5\,nm$ gold layer. The computation results show good agreement with physical experiments described in [13].

Note that with this method an arbitrary number of anisotropic layers can be accounted. The limiting material describing by formula (5.6) can also be treated because the integral term in (5.6) is negligible due to very strong time decay of the function ω.

7 Numerical simulation

Using the derived model, we simulate a surface acoustic wave sensor (compare with the introduction) based on the multi-layered structure shown in Figure 6. The molecular layer adhering to the surface of the auxiliary gold layer is being modeled through the homogenization technique developed in this paper. The molecular layer is expected being well described as a new material associated with equation (5.6). Equations (5.4) and (5.5) describe then the overlying fluid and the underlying gold layer. Such a model allows us to compute the dispersal relation and the sensitivity (see the previous section).

Figure 7 presents computed graphs of the sensitivity versus the guiding layer thickness. The loading is modeled through doubling the thickness of the homogenized layer. The thickness of the gold layer is being varied from $0.5\,nm$ to $300\,nm$. The best sensitivity is achieved at $200\,nm$. Above this value, decreasing the sensitivity is observed. The light curve in Figure 7 corresponds to the gold layer thickness of $300\,nm$. The computation results are consistent with physical experiments described in [13]. The sensitivity is greater in our case because the additional mass loading is being modeled through a bristle layer whose resistance is greater than the one of the gold layer used in [13] as the mass loading.

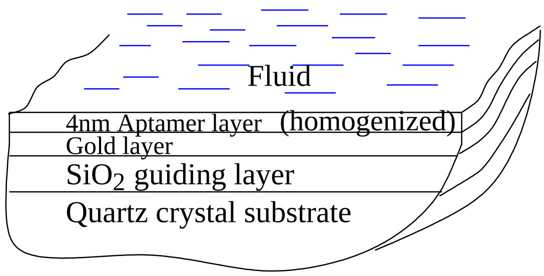

Figure 6: Multi-layered structure.

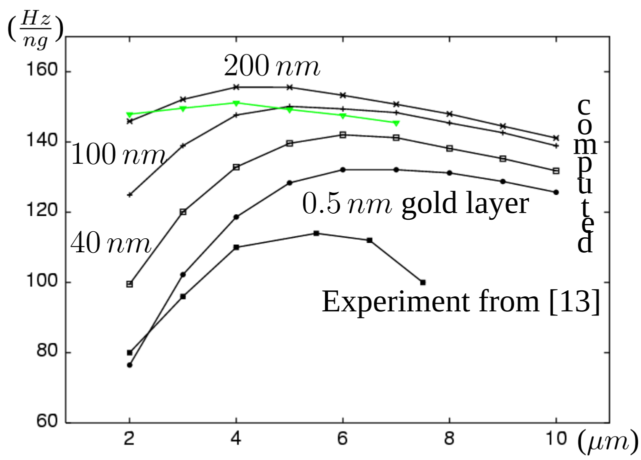

Figure 7: Comparison of numerically computed and experimentally measured sensitivities.

The next simulation (Figure 8) shows the sensitivity of the sensor regarding an additional homogenized protein layer adhering to the aptamer layer (compare with Figure 6). The dependence on the aptamer packing density is presented. Thereby, the protein packing density is changing so that their ratio remains constant. The packing density is defined as $|\Sigma_S|/(|\Sigma_S| + |\Sigma_F|)$, see Figure 2. The thicknesses of the aptamer and protein layers are 21A and 43A, respectively.

Figure 9 represents the time performance of the etching process. In this case, a $9\,nm$ copper layer was used instead of the molecular one. The water flux is being alternated with the flux of an acid solution that etches the copper layer. The phase shift is being measured. A step at the acid-to-water transition is caused by the change of the fluid viscosity. The results are in a good agreement with the measurements [14] done in c a e s a r laboratories.

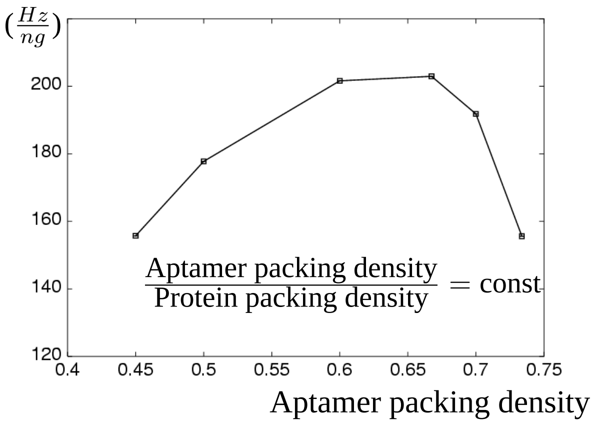

Figure 8: Sensitivity for various packing densities.

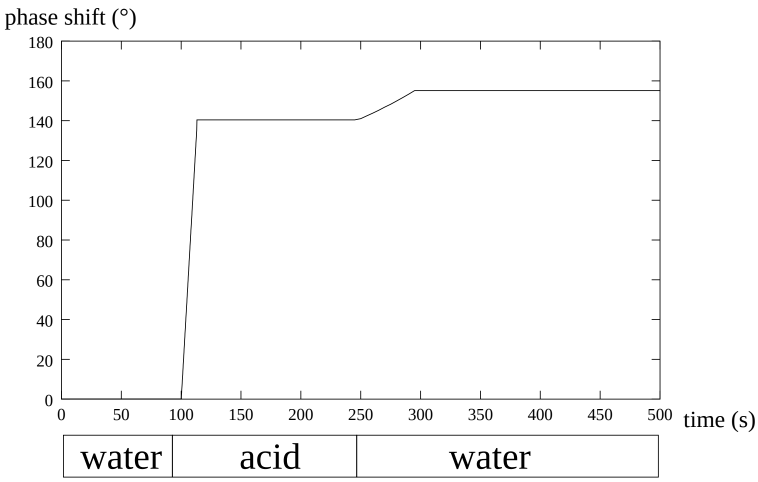

Figure 9: Modeling of the time behavior of the phase shift when etching a copper layer with an acid.

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