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Oscillations of Nonlinear Thin Plates Excited by Piezoelectric Patches

A model describing oscillations of nonlinear thin plates excited by patches made of piezoelectric ceramics is proposed. This model is the Kármán system with discontinuous coefficients and additional terms related to piezoelectric properties of patches. Such a model is applicable to the case where the bending is much less than the longitudinal dimension of the plate. The existence of solutions of nonlinear equations describing such composite plates is stated.

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0. Introduction

In this paper, oscillations of nonlinear thin plates excited by patches made of piezoelectric ceramics are considered. The aim consists in obtaining partial differential equations describing the phenomena and in the statement of the existence of their solutions. Linear models describing plates with patches made of piezoelectric ceramics are well known and were investigated in [1] in details. Such models are applicable to the case of extremely small bending that should be much less than the thickness of the plate. The range of applicability of the model proposed is restricted by the condition that the bending is much less than the longitudinal dimension of the plate (see [2]). Note that our model is the Kármán system [6] with discontinuous coefficients and additional terms related to piezoelectric properties of patches. These additional terms and discontinuous coefficients hinder a direct application of results of [6] to the considered model.

1. Models of piezoelectric media

We consider a macroscopic and quasi-static model of piezoelectric media. The first characteristic means that the model contains only mean values of physical magnitudes. The second characteristic assumes that the frequency of electric fields is sufficiently small so that magnetic fields do not appear, and electro-magnetic waves are absent (see [4] for general models).

According to the macroscopic approach to polarization the equation

$$P_i = D_i - \epsilon_0 E_i$$

describes the relationship between intensity of the electric field E , electric displacement D , and polarization P . The symbol ϵ_0 is the permittivity of free space.

Assume that the deformations of the medium are sufficiently small and use the following conventional form of the strain tensor,

$$d_{lm} := 1/2(u_{l,m} + u_{m,l} + u_{k,l} \cdot u_{k,m}),$$

where $u_l = y_l(x_m, t) - x_l$ is the displacement. Summation over repeated indices is assumed, and commas before indices denote differentiation with respect to the components of the vector x . Let σ_{ij} be the stress tensor. We consider linear material laws (see [5]),

$$\sigma_{ij} = C_{ijkl}d_{kl} - e_{kij}E_k, \quad D_i = \epsilon_{ij}E_j - e_{ikl}d_{kl}.$$

The coefficients are such that C_{ijkl} is the elastic tensor, e_{ikl} is the piezoelectric stress tensor, and ϵ_{ij} is the permittivity tensor.

It is inconvenient to operate with tensors of the fourth rank. To avoid this, one applies the Voigt notation (see e.g. [5]):

$$\sigma_{ij} = \sigma_\lambda, \quad d_{ij} = d_\lambda, \quad C_{ijkl} = C_{\lambda\mu}, \quad e_{ikl} = e_{i\lambda}, \quad i = 1, 2, 3; \quad \lambda, \mu = 1, \dots, 6.$$

That is, one replaces the couples (ij) , (kl) by the indices λ, μ using the correspondence given in Table 1. With this new notation, the material laws look like

$$\sigma_\lambda = C_{\lambda\mu}d_\mu - e_{i\lambda}E_i, \quad D_i = \epsilon_{ij}E_j - e_{i\lambda}d_\lambda.$$

Table 1

(i, j) and (k, l)	1, 1	2, 2	3, 3	2, 3 = 3, 2	1, 3 = 3, 1	1, 2 = 2, 1
λ and μ	1	2	3	4	5	6

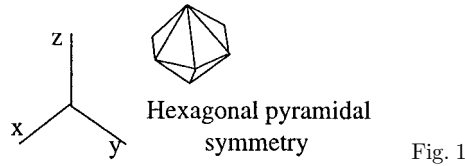


Fig. 1

Therefore, a piezoelectric material is characterized by the matrix

$$\begin{pmatrix} C_{\lambda\mu} & e'_{i\lambda} \\ e_{i\lambda} & \epsilon_{ij} \end{pmatrix},$$

which is called the *material matrix*. For example, the material matrix of a piezoelectric ceramic with hexagonal and pyramidal symmetry (see Figure 1) has the following form (see [5]):

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ c_{12} & c_{33} & c_{13} & 0 & 0 & 0 & 0 & 0 & e_{31} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & e_{33} \\ 0 & 0 & 0 & c_{44} & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 & e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \epsilon_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \epsilon_{33} \end{pmatrix}.$$

Additionally, there holds: $c_{12} \approx c_{13}$, $c_{11} \approx c_{33}$, $c_{44} \approx c_{33} - c_{13}$. Therefore, the elastic properties of piezoelectric ceramic are near to those of homogeneous media, and can be characterized by the Young elastic modulus $R = (c_{11} - c_{12})(c_{11} + 2c_{12}) / (c_{11} + c_{12})$, and the Poisson ratio $\sigma = c_{12} / (c_{11} + c_{12})$.

In conclusion of this section, we give the formula for the density of the free energy in the case of linear material laws (see e.g. [4]):

$$\chi = \frac{1}{2} (\sigma_{ij}d_{ij} - E_iP_i) = \frac{1}{2} C_{ijkl}d_{ij}d_{kl} - e_{ikl}d_{kl}E_i - \frac{1}{2} (\epsilon_{ij} - \epsilon_0\delta_{ij}) E_iE_j.$$

For piezoelectric ceramics, one obtains

$$\begin{aligned} \chi = \frac{R}{2(1+\sigma)} \left(d_{ij}^2 + \frac{\sigma}{1-2\sigma} d_{ll}^2 \right) - e_{15}d_{13}E_1 - e_{15}d_{13}E_2 - (e_{31}d_{11} + e_{31}d_{22} + e_{33}d_{33}) E_3 \\ - \frac{1}{2} [(\epsilon_{11} - \epsilon_0) E_1^2 + (\epsilon_{11} - \epsilon_0) E_2^2 + (\epsilon_{33} - \epsilon_0) E_3^2]. \end{aligned}$$

2. Oscillations of plates

2.1 Nonlinear plate with piezo-actuators

Without any loss of generality we consider a thin plate supplied by a single patch made of a piezoelectric ceramic. The plate itself consists of a metal, and the upper surface of the patch is covered by a metal. Therefore, the voltage can be applied to the patch as shown in Figure 2.

As usual, we assume the existence of a neutral surface on which all deformations caused by the “pure bending” are equal to zero (see Figure 3). Note that the deformations caused by the stretching of the plate do not vanish on the neutral surface. We discuss these two types of deformations below.

The state of the plate is defined by *three displacements*: ξ is the *vertical displacement*, u_1 and u_2 are *longitudinal displacements*.

From the boundary condition $\sigma_{ij}n_j = 0$, where n_j is the vertical normal to the surface of the plate, we get $\sigma_{i3} = 0$, $i = 1, 2, 3$. This yields the following relations between the components of the strain tensor:

$$d_{13} = 0, \quad d_{23} = 0, \quad d_{33} = -\frac{\sigma}{1-\sigma} (d_{11} + d_{22}). \tag{1}$$

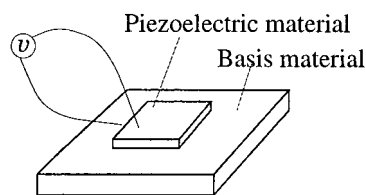


Fig. 2

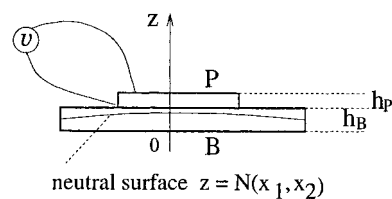


Fig. 3

Because of the Maxwell equation

$$\mathbf{rot} E = 0,$$

the jump condition

$$n \times [E] = 0, \quad z = h_B, \quad z = h_B + h_P$$

holds. Hence,

$$E_x = E_y \approx 0, \quad E_z \approx \frac{v}{h_P}, \quad z \in (h_B, h_B + h_P).$$

As it was shown in [2], one can separately consider the deformation caused by the bending when the upper layers of the plate are being pressed (stretched) whereas the low layers are stretched (pressed) and the deformation related to the stretching (pressing) of the plate as if it were a film of a zero thickness.

In the case of deformation of the first type, one can express non-zero components of the strain tensor via the vertical displacement using the neutral surface:

$$d_{11} = -(z - N(x_1, x_2)) \frac{\partial^2 \xi}{\partial x_1^2}, \quad d_{22} = -(z - N(x_1, x_2)) \frac{\partial^2 \xi}{\partial x_2^2}, \quad d_{12} = -(z - N(x_1, x_2)) \frac{\partial^2 \xi}{\partial x_1 \partial x_2}.$$

The other components are defined by (1), where σ is either σ_P or σ_B depending on the considered material.

The deformation of the second kind is described by the strain tensor the components of which are

$$d_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial \xi}{\partial x_\alpha} \frac{\partial \xi}{\partial x_\beta} \right), \quad \alpha, \beta = 1, 2, \tag{2}$$

and the other components are defined by (1), where σ is either σ_P or σ_B . The particular tensor $d_{\alpha\beta}$, $\alpha, \beta = 1, 2$ is called *plain strain tensor*.

Let us introduce the following notation:

$$\begin{aligned} \tau_{11}^P &= \frac{R_P}{1 - \sigma_P^2} (d_{11} + \sigma_P d_{22}); & \tau_{11}^B &= \frac{R_B}{1 - \sigma_B^2} (d_{11} + \sigma_B d_{22}); \\ \tau_{22}^P &= \frac{R_P}{1 - \sigma_P^2} (d_{22} + \sigma_P d_{11}); & \tau_{22}^B &= \frac{R_B}{1 - \sigma_B^2} (d_{22} + \sigma_B d_{11}); \\ \tau_{12}^P &= \frac{R_P}{1 + \sigma_P} d_{12}; & \tau_{12}^B &= \frac{R_B}{1 + \sigma_B} d_{12}. \end{aligned} \tag{3}$$

The *density of the free energy of the piezoelectric material* is defined as follows (see [2]):

$$\begin{aligned} \chi_P &= (z - N(x_1, x_2))^2 \frac{R_P}{1 + \sigma_P} \left\{ \frac{1}{2(1 - \sigma_P)} (\Delta \xi)^2 + (\xi_{x_1 x_2})^2 - \xi_{x_1 x_1} \xi_{x_2 x_2} \right\} + \tau_{\alpha\beta}^P d_{\alpha\beta} \\ &\quad - (z - N(x_1, x_2)) \left(\frac{e_{33} \sigma_P}{1 - \sigma_P} - e_{31} \right) \Delta \xi E_z - \left(\frac{e_{33} \sigma_P}{1 - \sigma_P} - e_{31} \right) (d_{11} + d_{22}) E_z - \frac{1}{2} (\epsilon_{33} - 1) E_z^2. \end{aligned}$$

The *density of the free energy of the basis material* is

$$\chi_B = (z - N(x_1, x_2))^2 \frac{R_B}{1 + \sigma_B} \left\{ \frac{1}{2(1 - \sigma_B)} (\Delta \xi)^2 + (\xi_{x_1 x_2})^2 - \xi_{x_1 x_1} \xi_{x_2 x_2} \right\} + \tau_{\alpha\beta}^B d_{\alpha\beta}.$$

The whole *free energy* is obtained via integration over the volumes of the piezoelectric and basis materials:

$$F = \iiint_P \chi_P + \iiint_B \chi_B.$$

The *variational principle* reads:

$$\begin{aligned} \delta_\xi F + \iiint_P \varrho_P \xi_{tt} \delta \xi + \iiint_B \varrho_B \xi_{tt} \delta \xi + \delta_{u_1} F + \iiint_P \varrho_P u_{1tt} \delta u_1 + \iiint_B \varrho_B u_{1tt} \delta u_1 \\ + \delta_{u_2} F + \iiint_P \varrho_P u_{2tt} \delta u_2 + \iiint_B \varrho_B u_{2tt} \delta u_2 = 0. \end{aligned}$$

Integration over z and putting together the integrals over S_P yields a weak formulation of the system:

$$\begin{aligned} \iint_{S_P} (\tilde{\varrho}_P \xi_{tt} \varphi + \tilde{\gamma}_P \Delta \xi \Delta \varphi + \tilde{\tau}_{\alpha\beta}^P \xi_{x_\beta} \varphi_{x_\alpha} - v(t) K \Delta \varphi + v(t) G \xi_{x_\alpha} \varphi_{x_\alpha}) + \iint_{S_B} (\varrho_B \xi_{tt} \varphi + \gamma_B \Delta \xi \Delta \varphi + \tau_{\alpha\beta}^B \xi_{x_\beta} \varphi_{x_\alpha}) \\ + \iint_{S_P} (\tilde{\varrho}_P u_{1tt} \psi_1 + \tilde{\tau}_{1\beta}^P \psi_{1x_\beta} + v(t) G \psi_{1x_1}) + \iint_{S_B} (\varrho_B u_{1tt} \psi_1 + \tau_{1\beta}^B \psi_{1x_\beta}) \\ + \iint_{S_P} (\tilde{\varrho}_P u_{2tt} \psi_2 + \tilde{\tau}_{2\beta}^P \psi_{2x_\beta} + v(t) G \psi_{2x_2}) + \iint_{S_B} (\varrho_B u_{2tt} \psi_2 + \tau_{2\beta}^B \psi_{2x_\beta}) = 0. \end{aligned} \tag{4}$$

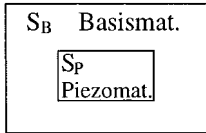


Fig. 4

Here the variations $\delta\xi, \delta u_1, \delta u_2$ were replaced by φ, ψ_1, ψ_2 from $H_0^2(S), H_0^1(S), H_0^1(S)$, respectively. The regions S_P, S_B , and $S = S_P \cup S_B$ are shown in Figure 4. Note that S_B is the complement of S_P .

Using the integration by parts, we arrive at the classical formulation of the plate equations,

$$\begin{aligned} \varrho \xi_{tt} + \operatorname{div} \nabla(\gamma \Delta \xi) - \frac{\partial}{\partial x_\alpha} (\tau_{\alpha\beta} \xi_{x_\beta}) &= v(t) \Delta(K \cdot I_{S_P}) + v(t) G \frac{\partial}{\partial x_\alpha} (\xi_{x_\alpha} I_{S_P}), \\ \varrho u_{1tt} - \frac{\partial}{\partial x_\beta} \tau_{1\beta} &= +v(t) G \frac{\partial}{\partial x_1} I_{S_P}, \quad \varrho u_{2tt} - \frac{\partial}{\partial x_\beta} \tau_{2\beta} = +v(t) G \frac{\partial}{\partial x_2} I_{S_P}. \end{aligned} \tag{5}$$

Here I_{S_P} is the indicator function of S_P . Boundary and initial conditions are:

$$\begin{aligned} \xi|_G = 0, \quad \frac{\partial \xi}{\partial \mathbf{n}} \Big|_G &= 0, \quad \xi|_{t=0} = \xi_0, \quad \xi_t|_{t=0} = \xi'_0, \\ u_\alpha|_G = 0, \quad u_\alpha|_{t=0} &= u_{\alpha 0}, \quad u_{\alpha t}|_{t=0} = u'_{\alpha 0}, \quad \alpha = 1, 2. \end{aligned} \tag{6}$$

The following additional interface conditions arise from the integration by parts:

$$\begin{aligned} [\xi] = 0, \quad \left[\frac{\partial \xi}{\partial \mathbf{n}} \right] &= 0, \quad [\gamma \Delta \xi] = 0, \quad \left[\frac{\partial}{\partial \mathbf{n}} \gamma \Delta \xi \right] = 0, \\ [u_\alpha] = 0, \quad [\tau_{\alpha\beta} \cdot n_\beta] &= 0, \quad \alpha = 1, 2. \end{aligned} \tag{7}$$

Here $[\]$ denotes the jump of a function on the boundary between S_P and S_B .

The coefficients in (5) are discontinuous functions defined by the formulas

$$\varrho = \begin{cases} \tilde{\varrho}_P, & (x_1, x_2) \in S_P, \\ \varrho_B, & (x_1, x_2) \in S_B, \end{cases} \quad \gamma = \begin{cases} \tilde{\gamma}_P(x_1, x_2), & (x_1, x_2) \in S_P, \\ \gamma_B(x_1, x_2), & (x_1, x_2) \in S_B, \end{cases} \tag{8}$$

$$\tau_{\alpha\beta} = \begin{cases} \tilde{\tau}_{\alpha\beta}^P, & (x_1, x_2) \in S_P, \\ \tau_{\alpha\beta}^B, & (x_1, x_2) \in S_B. \end{cases} \tag{9}$$

Here

$$\begin{aligned} \tilde{\varrho}_P &= \varrho_B + \varrho_P \frac{h_P}{h_B}; \\ \tilde{\gamma}_P(x_1, x_2) &= \frac{R_B}{2h_B(1 - \sigma_B^2)} \int_0^{h_B} (z - N(x_1, x_2))^2 dz + \frac{R_P}{2h_B(1 - \sigma_P^2)} \int_0^{h_P} (z + h_B - N(x_1, x_2))^2 dz; \\ \gamma_B(x_1, x_2) &= \frac{R_B}{2h_B(1 - \sigma_B^2)} \int_0^{h_B} (z - N(x_1, x_2))^2 dz; \\ K(x_1, x_2) &= \frac{1}{h_P h_B} \left(\frac{e_{33} \sigma_P}{1 - \sigma_P^2} - e_{13} \right) \int_0^{h_P} (z + h_B - N(x_1, x_2))^2 dz; \\ G &= \frac{1}{h_P h_B} \left(\frac{e_{33} \sigma_P}{1 - \sigma_P^2} - e_{13} \right); \\ \tilde{\tau}_{11}^P &= \frac{\tilde{R}_P}{1 - \tilde{\sigma}_P^2} (d_{11} + \tilde{\sigma}_P d_{22}); \quad \tilde{\tau}_{22}^P = \frac{\tilde{R}_P}{1 - \tilde{\sigma}_P^2} (d_{22} + \tilde{\sigma}_P d_{11}); \quad \tilde{\tau}_{12}^P = \frac{\tilde{R}_P}{1 + \tilde{\sigma}_P} d_{12}, \end{aligned} \tag{10}$$

where

$$\tilde{R}_P = \frac{a_1^2 - a_2^2}{a_1}, \quad \tilde{\sigma}_P = \frac{a_2}{a_1}, \quad a_1 = \frac{R_B}{1 - \sigma_B^2} + \frac{R_P}{1 - \sigma_P^2} \frac{h_P}{h_B}, \quad a_2 = \frac{R_B \sigma_B}{1 - \sigma_B^2} + \frac{R_P \sigma_P}{1 - \sigma_P^2} \frac{h_P}{h_B}.$$

Note that the coefficients $\tilde{\varrho}_P, \varrho_B, G$, and the coefficients defining $\tilde{\tau}_{\alpha\beta}^P$ and $\tau_{\alpha\beta}^B$ are constant. The functions $\tilde{\gamma}_P, \gamma_B$, and K are defined by the neutral surface. One can easily verify that

$$0 < \underline{\gamma} \leq \gamma(x_1, x_2) \leq \bar{\gamma}, \quad 0 < \underline{\varrho} \leq \varrho \leq \bar{\varrho}, \quad \tau_{\alpha\beta} d_{\alpha\beta} \geq \nu d_{\alpha\beta} d_{\alpha\beta} \tag{11}$$

with some positive constants $\underline{\gamma}, \bar{\gamma}, \underline{\varrho}, \bar{\varrho}$, and ν .

2.2 Existence of solutions

We say that functions

$$\xi \in H^1(0, T; L_2(S)) \cap L_2(0, T; H_0^2(S)), \quad u_\alpha \in H^1(0, T; L_2(S)) \cap L_2(0, T; H_0^1(S)), \quad \alpha = 1, 2,$$

satisfying the initial conditions

$$\xi(0, \cdot) = \xi_0, \quad u_\alpha(0, \cdot) = u_{\alpha 0}, \quad \alpha = 1, 2,$$

form a *generalized solution to system (5)–(7)* if the following equality holds:

$$\begin{aligned} & \int_0^T \int \int (-\varrho \xi_t \varphi_t + \gamma \Delta \xi \Delta \varphi + \tau_{\alpha\beta} \xi_{x_\beta} \varphi_{x_\alpha}) \, dx \, dt - \int \int \varrho \xi_0 \varphi(0, x) \, dx \\ & + \int_0^T \int \int (-v(t) K \Delta \varphi + v(t) G \xi_{x_\alpha} \varphi_{x_\alpha}) \, dx \, dt + \int_0^T \int \int (-\varrho u_{\alpha t} \psi_{\alpha t} + \tau_{\alpha\beta} \psi_{\alpha x_\beta}) \, dx \, dt \\ & - \int \int \varrho u'_{\alpha 0} \psi_\alpha(0, x) \, dx + \int_0^T \int \int v(t) G \psi_{\alpha x_\alpha} \, dx \, dt = 0 \end{aligned} \tag{12}$$

$$\text{for all } \varphi \in H_T^1(0, T; L_2(S)) \cap L_2(0, T; H_0^2(S)), \quad \psi_\alpha \in H_T^1(0, T; L_2(S)) \cap L_2(0, T; H_0^1(S)), \quad \alpha = 1, 2,$$

where the index T points out to the additional conditions: $\varphi(T, \cdot) = 0, \psi_\alpha(T, \cdot) = 0$.

Theorem: *Let $\xi_0 \in H_0^2(S), \xi'_0 \in L_2(S), u_{\alpha 0} \in H_0^1(S), u'_{\alpha 0} \in L_2(S), \alpha = 1, 2$, and $v(\cdot) \in H^1(0, T)$. Then the system (5)–(7) has a solution such that*

$$(\xi, \xi_t) \in L_\infty(0, T; H_0^2(S)) \times L_\infty(0, T; L_2(S)), \quad (u_\alpha, u_{\alpha t}) \in L_\infty(0, T; H_0^1(S)) \times L_\infty(0, T; L_2(S)).$$

Proof: Let $\{\omega_i\}_{i=1}^\infty$ be a basis of $H_0^2(S)$ which is orthonormal in $L_2(S)$. Similarly, let $\{\eta_i\}_{i=1}^\infty$ be a basis of $H_0^1(S)$ which is orthonormal in $L_2(S)$. Consider Galerkin approximations of the form

$$\xi^m = \sum_{i=0}^m a_i^m(t) \omega_i, \quad u_\alpha^m = \sum_{i=0}^m b_{\alpha i}^m(t) \eta_i, \quad \alpha = 1, 2, \tag{13}$$

where $a_i^m(t), b_{\alpha i}^m(t)$ are unknown functions. We substitute approximations (13) in (12) together with test functions of the form $\varphi(t) \omega_j, \psi_\alpha(t) \eta_j, \alpha = 1, 2, j = 1, \dots, m$, where $\varphi(t), \psi_\alpha(t)$ are arbitrary functions from $C_T^1[0, T]$, where the index T means that $\varphi(T) = 0, \psi_\alpha(T) = 0$. Using an arbitrary choice of $\varphi(\cdot), \psi_\alpha(\cdot)$, we obtain the following system of ordinary differential equations defining the functions $a_i^m(t), b_{\alpha i}^m(t), \alpha = 1, 2, j = 1, \dots, m$:

$$\begin{aligned} \ddot{a}_j^m(t) + A_j^m(t, a_1^m, \dots, a_m^m, b_{11}^m, \dots, b_{1m}^m, b_{21}^m, \dots, b_{2m}^m) &= 0, \\ \dot{b}_{1j}^m(t) + B_{1j}^m(t, a_1^m, \dots, a_m^m, b_{11}^m, \dots, b_{1m}^m, b_{21}^m, \dots, b_{2m}^m) &= 0, \\ \dot{b}_{2j}^m(t) + B_{2j}^m(t, a_1^m, \dots, a_m^m, b_{11}^m, \dots, b_{1m}^m, b_{21}^m, \dots, b_{2m}^m) &= 0. \end{aligned} \tag{14}$$

With the initial conditions:

$$\begin{aligned} a_j^m(0) &= (P_{H_0^2(S)}^m \xi_0)_j, & \dot{a}_j^m(0) &= (\xi'_0, \omega_j)_{L_2(S)}, \\ b_{1j}^m(0) &= (P_{H_0^1(S)}^m u_{10})_j, & \dot{b}_{1j}^m(0) &= (u'_{10}, \omega_j)_{L_2(S)}, \\ b_{2j}^m(0) &= (P_{H_0^1(S)}^m u_{20})_j, & \dot{b}_{2j}^m(0) &= (u'_{20}, \omega_j)_{L_2(S)}. \end{aligned} \tag{15}$$

Here $(P_{H_0^2(S)}^m \xi_0)_j$ is the j -th component of the $H_0^2(S)$ projection of ξ_0 onto $\text{span}\{\omega_1, \dots, \omega_m\}$. Similarly, $(P_{H_0^1(S)}^m u_{\alpha 0})_j$ is the j -th component of the $H_0^1(S)$ projection of $u_{\alpha 0}$ onto $\text{span}\{\eta_1, \dots, \eta_m\}$. The functions A_i^m, B_{1i}^m and B_{2i}^m are defined in the usual way. Note that they are continuous with respect to their variables for each fixed m . Later we estimate the approximate solutions $\xi^m, u_\alpha^m, \alpha = 1, 2$, and obtain in particular that $\max_{t \in [0, T]} \|\xi^m(t, \cdot)\|_{L_2(S)} \leq C$, and $\max_{t \in [0, T]} \|u_\alpha^m(t, \cdot)\|_{L_2(S)} \leq C$, where C is an independent from m constant, and $[0, T]$ is the maximal time interval on which the solution of (14) exists. Due to the relations $\|\xi^m(t, \cdot)\|_{L_2(S)}^2 = \sum_{i=1}^m a_i^{m2}(t)$ and $\|u_\alpha^m(t, \cdot)\|_{L_2(S)}^2 = \sum_{i=1}^m b_{\alpha i}^{m2}(t)$, one can conclude that the solutions of (14) are bounded and, hence, can be extended to the arbitrary time interval $[0, T]$.

Let us go on to the estimation of ξ^m, u_1^m , and u_2^m . We substitute these functions into (12), test (12) with the test functions $\varphi(t) a_j^m(t) \omega_j, \psi(t) b_{\alpha j}^m(t) \eta_j, \alpha = 1, 2$, and take the sum over j from 1 to m . It is assumed that the functions $\varphi, \psi \in C_T^1[0, T]$ so that the test functions are admissible. Let us agree to omit the upper index m indicating the number of the basis functions when constructing the approximations ξ^m, u_1^m , and u_2^m . Then, taking into account the arbi-

bitrary choice of φ, ψ , we obtain:

$$\begin{aligned} & \int_0^t \int_S (\varrho \xi_{tt} \xi_t + \gamma \Delta \xi \Delta \xi_t + \tau_{\alpha\beta} \xi_{x_\beta} \xi_{x_\alpha t}) \, dx \, dt - \int_S \varrho (\xi'_0 - \xi_t(0, \cdot)) \xi_t(0, \cdot) \, dx \\ & + \int_0^t \int_{S_p} (-v(t) K \Delta \xi_t + v(t) G \xi_{x_\alpha} \xi_{x_\alpha t}) \, dx \, dt + \int_0^t \int_S (\varrho u_{att} u_{at} + \tau_{\alpha\beta} u_{\alpha\beta t}) \, dx \, dt \\ & - \int_S \varrho (u'_{\alpha 0} - u_{at}(0, \cdot)) u_{at}(0, \cdot) \, dx + \int_0^t \int_{S_p} v(t) G u_{\alpha x_\alpha t} \, dx \, dt = 0 \end{aligned}$$

for all $t \in [0, T]$. Then

$$\begin{aligned} & \int_0^t \frac{d}{dt} \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \tau_{\alpha\beta} d_{\alpha\beta}) \, dx \, dt \\ & - \int_0^t \int_{S_p} (v(t) K \Delta \xi_t - v(t) G \xi_{x_\alpha} \xi_{x_\alpha t} - v(t) G u_{\alpha x_\alpha t}) \, dx \, dt \\ & - \int_S [\varrho (\xi'_0 - \xi_t(0, \cdot)) \xi_t(0, \cdot) + \varrho (u'_{\alpha 0} - u_{at}(0, \cdot)) u_{at}(0, \cdot)] \, dx = 0. \end{aligned}$$

The last term tends to zero with growing m because of the choice of the initial conditions (15). Therefore, they are bounded, and hence

$$\begin{aligned} & \int_0^t \frac{d}{dt} \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \tau_{\alpha\beta} d_{\alpha\beta}) \, dx \, dt \\ & \leq \int_0^t \int_{S_p} (v(t) K \Delta \xi_t - v(t) G \xi_{x_\alpha} \xi_{x_\alpha t} - v(t) G u_{\alpha x_\alpha t}) \, dx \, dt + C. \end{aligned}$$

This implies

$$\begin{aligned} & \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \tau_{\alpha\beta} d_{\alpha\beta}) \, dx \\ & \leq \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \tau_{\alpha\beta} d_{\alpha\beta})|_{t=0} \, dx \\ & + \int_0^t \int_{S_p} (v(t) K \Delta \xi_t - v(t) G \xi_{x_\alpha} \xi_{x_\alpha t} - v(t) G u_{\alpha x_\alpha t}) \, dx \, dt + C. \end{aligned}$$

The first term on the right-hand-side is bounded because of (15). Hence,

$$\int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \tau_{\alpha\beta} d_{\alpha\beta}) \, dx \leq \int_0^t \int_{S_p} (v(t) K \Delta \xi_t - v(t) G d_{\alpha\alpha t}) \, dx \, dt + C \tag{17}$$

for all $t \in [0, T]$. We have used the equality $\xi_{x_\alpha} \xi_{x_\alpha t} + u_{\alpha x_\alpha t} = d_{\alpha\alpha t}$ (remember that summation over repeated indices is assumed). Using (11), we obtain

$$\int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \nu d_{\alpha\beta} d_{\alpha\beta}) \, dx \leq \int_0^t \int_{S_p} (v(t) K \Delta \xi_t - v(t) G d_{\alpha\alpha t}) \, dx \, dt + C. \tag{18}$$

From (18) we have

$$\begin{aligned} & \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \nu d_{\alpha\beta} d_{\alpha\beta}) \, dx \\ & \leq \int_{S_p} (v(t) K \Delta \xi - v(t) G d_{\alpha\alpha}) \, dx - \int_{S_p} (v(t) K \Delta \xi - v(t) G d_{\alpha\alpha})|_{t=0} \, dx \\ & + \int_0^t \int_{S_p} (v'(t) K \Delta \xi - v'(t) G d_{\alpha\alpha}) \, dx \, dt + C. \end{aligned}$$

Since the second term on the right-hand-side is bounded due to the initial conditions (15), we obtain:

$$\begin{aligned} & \int_S (\varrho (\xi_t)^2 + \gamma (\Delta \xi)^2 + \varrho (u_{at})^2 + \nu d_{\alpha\beta} d_{\alpha\beta}) \, dx \\ & \leq \int_{S_p} (v(t) K \Delta \xi - v(t) G d_{\alpha\alpha}) \, dx + \int_0^t \int_{S_p} (v'(t) K \Delta \xi - v'(t) G d_{\alpha\alpha}) \, dx \, dt + C. \end{aligned} \tag{19}$$

Due to the Young inequality, we have the following estimates:

$$\begin{aligned} \left| \iint_{S_P} v(t) K \Delta \xi \, dx \right| &\leq \frac{1}{2\varepsilon} v^2(t) \iint_{S_P} K^2 \, dx + \frac{\varepsilon}{2} \iint_S (\Delta \xi)^2 \, dx, \\ \left| \iint_{S_P} v(t) G d_{\alpha\alpha} \, dx \right| &\leq G^2 \frac{\text{meas}(S_P)}{2\varepsilon} v^2(t) + \frac{\varepsilon}{2} \iint_S d_{\alpha\beta} d_{\alpha\beta} \, dx, \\ \left| \int_0^t \iint_{S_P} v(t)' K \Delta \xi \, dx \, dt \right| &\leq \frac{1}{2\varepsilon} \int_0^t (v'(t))^2 \, dt \iint_{S_P} K^2 \, dx + \frac{\varepsilon}{2} \int_0^t \iint_S (\Delta \xi)^2 \, dx, \\ \left| \int_0^t \iint_{S_P} v'(t) G d_{\alpha\alpha} \, dx \, dt \right| &\leq G^2 \frac{\text{meas}(S_P)}{2\varepsilon} \int_0^t (v'(t))^2 \, dt + \frac{\varepsilon}{2} \int_0^t \iint_S d_{\alpha\beta} d_{\alpha\beta} \, dx \, dt, \end{aligned}$$

where ε is an arbitrary positive value.

Taking this into account, we obtain

$$\iint_S ((\xi_t)^2 + (\Delta \xi)^2 + u_{\alpha t} u_{\alpha t} + d_{\alpha\beta} d_{\alpha\beta}) \, dx \leq \mu \int_0^t \iint_S ((\xi_t)^2 + (\Delta \xi)^2 + u_{\alpha t} u_{\alpha t} + d_{\alpha\beta} d_{\alpha\beta}) \, dx \, dt + C. \tag{20}$$

Applying the Gronwall-Lemma yields

$$\iint_S ((\xi_t)^2 + (\Delta \xi)^2 + u_{\alpha t} u_{\alpha t} + d_{\alpha\beta} d_{\alpha\beta}) \, dx \leq C \tag{21}$$

for any $t \in [0, T]$, where C is independent of m .

Using the fact that (see [7])

$$C_1 \iint_S \xi_{x_\alpha x_\alpha}^2 \, dx \leq \iint_S (\Delta \xi)^2 \, dx \leq C_2 \iint_S \xi_{x_\alpha x_\alpha}^2 \, dx,$$

we obtain

$$\iint_S (\xi_t^2 + \xi_{x_\alpha x_\alpha} \xi_{x_\alpha x_\alpha} + u_{\alpha t} u_{\alpha t} + d_{\alpha\beta} d_{\alpha\beta}) \, dx \leq C. \tag{22}$$

From the definition of $d_{\alpha\beta}$ and the last estimate follows

$$\begin{aligned} \iint_S d_{\alpha\beta} d_{\alpha\beta} \, dx &= \frac{1}{4} \iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha} + \xi_{x_\alpha} \xi_{x_\beta})^2 \, dx \\ &= \frac{1}{4} \iint_S ((u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 + 2(u_{\alpha x_\beta} + u_{\beta x_\alpha}) \xi_{x_\alpha} \xi_{x_\beta} + (\xi_{x_\alpha} \xi_{x_\beta})^2) \, dx \leq C. \end{aligned}$$

Hence

$$\iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 \, dx \leq 4C - \iint_S (2(u_{\alpha x_\beta} + u_{\beta x_\alpha}) \xi_{x_\alpha} \xi_{x_\beta} + (\xi_{x_\alpha} \xi_{x_\beta})^2) \, dx.$$

Using the Young inequality, we obtain

$$\iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 \, dx \leq 4C + \varepsilon \iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 \, dx + \left(\frac{1}{\varepsilon} + 1\right) \iint_S (\xi_{x_\alpha} \xi_{x_\beta})^2 \, dx.$$

Taking into account (22) and the continuity of the embedding of $H_0^2(S)$ into $W^{1,q}(S)$, for any q , we conclude that the last term on the right-hand-side is bounded, and, hence,

$$\iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 \, dx \leq C.$$

Now, using the Korn's inequality (see e.g. [8]), we get

$$\iint_S u_{\alpha x_\beta} u_{\alpha x_\beta} \, dx \leq \lambda \iint_S (u_{\alpha x_\beta} + u_{\beta x_\alpha})^2 \, dx \leq C. \tag{23}$$

Taking into account the estimate (22), we obtain

$$\iint_S (\xi_t^2 + \xi_{x_\alpha x_\alpha} \xi_{x_\alpha x_\alpha} + u_{\alpha t} u_{\alpha t} + u_{\alpha x_\beta} u_{\alpha x_\beta}) \, dx \leq C$$

for all $t \in [0, T]$. Therefore (remember the index m), we may state that

$$\begin{aligned} \{\xi^m\} & \text{ is bounded in } L_\infty((0, T); H_0^2(S)), & \{\xi_t^m\} & \text{ is bounded in } L_\infty((0, T); L_2(S)), \\ \{u_\alpha^m\} & \text{ is bounded in } L_\infty((0, T); H_0^1(S)), & \{u_{\alpha t}^m\} & \text{ is bounded in } L_\infty((0, T); L_2(S)). \end{aligned} \tag{24}$$

From the compactness of the embedding $H_0^2(S) \subset W^{1,q}(S)$, for any $q > 1$, and from (24) we conclude that $\{\xi^m\}$ is relative compact in $C([0, T]; W^{1,q}(S))$ for any $q > 1$ (see [9]). So, we can assume that

$$\begin{aligned} \xi^m & \rightharpoonup \xi^0 \quad \text{*weakly in } L_\infty((0, T); H_0^2(S)), \\ \xi_t^m & \rightharpoonup \xi_t^0 \quad \text{*weakly in } L_\infty((0, T); L_2(S)), \\ \xi^m & \rightarrow \xi^0 \quad \text{in } C([0, T]; W^{1,q}(S)), \\ u_\alpha^m & \rightharpoonup u_\alpha^0 \quad \text{*weakly in } L_\infty((0, T); H_0^1(S)), \\ u_{\alpha t}^m & \rightharpoonup u_{\alpha t}^0 \quad \text{*weakly in } L_\infty((0, T); L_2(S)). \end{aligned} \tag{25}$$

Let us show that the functions $\xi^0, u_\alpha^0, \alpha = 1, 2$, form a solution of (5)–(7). Note that the sets of functions of the form $\varphi = x(t) \omega_j(x)$ and $\psi = y(t) \eta_j(x)$, where $x(\cdot), y(\cdot) \in C_T^1[0, T]$ are dense in $H_T^1(0, T; L_2(S)) \cap L_2(0, T; H_0^2(S))$ and $H_T^1(0, T; L_2(S)) \cap L_2(0, T; H_0^1(S))$, respectively (see [10]). Therefore, it is sufficient to prove that (12) holds for such functions along with $\xi^0, u_\alpha^0, \alpha = 1, 2$.

Obviously, ξ^m and $u_\alpha^m, \alpha = 1, 2$, satisfy (12) if m is larger than some integer depending on the choice of the test function. For linear terms, the passage to the limit in (12) is quite obvious. Consider nonlinear terms. These are

$$\int_0^T \int_S \tau_{\alpha\beta} \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt, \quad \int_0^T \int_S \tau_{\alpha\beta} \psi_{\alpha x_\beta} \, dx \, dt.$$

Taking into account the definition of $\tau_{\alpha\beta}$ (see (2), (3), (9), (10)), we obtain

$$\begin{aligned} \int_0^T \int_S \tau_{\alpha\beta} \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt & = \int_0^T \int_S C_{ija\beta} d_{ij} \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt = \int_0^T \int_S C_{ija\beta} (u_{ix_j}^m + u_{jx_i}^m + \xi_{x_i}^m \xi_{x_j}^m) \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt, \\ \int_0^T \int_S \tau_{\alpha\beta} \psi_{\alpha x_\beta} \, dx \, dt & = \int_0^T \int_S C_{ija\beta} d_{ij} \psi_{\alpha x_\beta} \, dx \, dt = \int_0^T \int_S C_{ija\beta} (u_{ix_j}^m + u_{jx_i}^m + \xi_{x_i}^m \xi_{x_j}^m) \psi_{\alpha x_\beta} \, dx \, dt, \end{aligned}$$

where $C_{ija\beta}$ are piecewise continuous functions being independent from t . Let us extract nonlinear terms again. These are terms of the form

$$\int_0^T \int_S u_{ix_j}^m \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt, \quad \int_0^T \int_S \xi_{x_i}^m \xi_{x_j}^m \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt, \quad \int_0^T \int_S \xi_{x_i}^m \xi_{x_j}^m \psi_{\alpha x_\beta} \, dx \, dt.$$

One can easily see from (25) that

$$\xi_{x_i}^m \xi_{x_j}^m \xi_{x_\beta}^m \rightarrow \xi_{x_i}^0 \xi_{x_j}^0 \xi_{x_\beta}^0 \quad \text{and} \quad \xi_{x_i}^m \xi_{x_j}^m \rightarrow \xi_{x_i}^0 \xi_{x_j}^0 \quad \text{in } L_2((0, T); L_2(S)),$$

and, hence, the last two of the nonlinear terms converge to the desired values. As for the first nonlinear term, we have

$$\int_0^T \int_S u_{ix_j}^m \xi_{x_\beta}^m \varphi_{x_\alpha} \, dx \, dt - \int_0^T \int_S u_{ix_j}^0 \xi_{x_\beta}^0 \varphi_{x_\alpha} \, dx \, dt = \int_0^T \int_S ((u_{ix_j}^m - u_{ix_j}^0) \xi_{x_\beta}^0 \varphi_{x_\alpha} + (\xi_{x_\beta}^m - \xi_{x_\beta}^0) u_{ix_j}^m \varphi_{x_\alpha}) \, dx \, dt.$$

Note that $\xi_{x_\beta}^0, \varphi_{x_\alpha} \in C([0, T]; L_q(S))$ for any $q > 1$ (see (25) and the definition of φ and take into account the embedding $H_0^2(S) \subset W^{1,q}(S)$).

Then, from the forth relation of (25), we conclude that

$$\int_0^T \int_S (u_{ix_j}^m - u_{ix_j}^0) \xi_{x_\beta}^0 \varphi_{x_\alpha} \, dx \, dt \rightarrow 0.$$

Moreover, we have

$$\left| \int_0^T \int_S (\xi_{x_\beta}^m - \xi_{x_\beta}^0) u_{ix_j}^m \varphi_{x_\alpha} \, dx \, dt \right| \leq \left[\int_0^T \int_S (u_{ix_j}^m \varphi_{x_\alpha})^2 \, dx \, dt \right]^{2/3} \left[\int_0^T \int_S (\xi_{x_\beta}^m - \xi_{x_\beta}^0)^3 \, dx \, dt \right]^{1/3}$$

and

$$\left[\int_0^T \int_S (u_{ix_j}^m \varphi_{x_\alpha})^2 \, dx \, dt \right] \leq \left[\int_0^T \int_S (u_{ix_j}^m)^2 \, dx \, dt \right]^{3/4} \left[\int_0^T \int_S (\varphi_{x_\alpha})^6 \, dx \, dt \right]^{1/4} \leq C.$$

Therefore, we conclude that

$$\int_0^T \iint_S (\xi_{x_\beta}^m - \xi_{x_\beta}^0) u_{ix_j}^m \varphi_{x_\alpha} dx dt \rightarrow 0.$$

This completes the proof. ■

3. Conclusion

We have proved the existence of solutions under the assumption that the control $dv(t)/dt$ is quadratic integrable. The uniqueness could also be easily proved under assumptions about some additional smoothness of solutions. It should be noted that such assumptions would not be realistic since the smoothness can not be improved because of the term $\Delta I_{S_p} \in H^{-2}(S)$ on the right-hand-side of (5) and the discontinuity of the coefficient γ .

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