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Homogenization of von Kármán Plates Excited by Piezoelectric Patches

A model describing vibration of nonlinear von Kármán thin plates excited by actuators made of piezoelectric ceramics is considered. The model contains strong oscillating coefficients due to the piezoelectric actuators. A procedure of homogenization based on the so-called two-scale convergence is applied to the model. This yields a nonlinear system of equations with constant coefficients. The unique solvability of the resulting system is proved. The convergence of all solutions of the original system to the solution of the resulting system as the number of piezoelectric actuators goes to infinity is proved.

Key words: nonlinear von Kármán thin plates, piezoelectric actuators, homogenization, two-scale convergence

MSC (1991): 35B27, 73K10, 73R05

1. Introduction

The problem of homogenization of partial differential equations describing vibration of nonlinear thin plates excited by actuators made of piezoelectric ceramics (see [1] and [2]) is considered. It is assumed that the number of the actuators goes to infinity whereas their dimension tends to zero. A procedure of homogenization based on the theory of two-scale convergence studied in [3, 4, 5, 6] is used. Specific features of the problem considered are: time dependent equations, the appearance of the forth spatial derivatives in the first equation describing vertical displacements of the plate, and nonlinearities typical for von Kármán systems. We apply a result of [6] about two-scale convergence of the second derivatives of subsequences of sequences bounded in $L_2(0,T;H_0^2(S))$, which enables us to handle a weak formulation of the problem. Results of [7] and [8] are used. Computer simulations demonstrate a good approximation of solutions of the original equations by solutions of the homogenized equations whenever the number of piezoelectric patches is sufficiently large.

2. Notation

$S \subset R^2$	is the domain occupied by the plate.
$K(t,x)$	is a prescribed distribution of the voltage over the whole plate S.
$S_{P_l} \subset S$	is the domain occupied by the lth piezopatch.
$S_P := \bigcup_{l=1}^{m} S_{P_l}$	is the domain occupied by all piezopatches.
$S_B := S \setminus S_P$	is the domain occupied by the base material.
$Y := [0,1] \times [0,1]$	is the unit square.
$\langle g \rangle := \iint_Y g(y)\, dy$	is the mean value of a function.
$C_\#^\infty(Y) \subset C^\infty(R^2)$	is the subspace of Yperiodic functions, i.e. $g(y_1+1,y_2) = g(y_1,y_2+1) = g(y_1,y_2)$, for all $(y_1,y_2) \in R^2$.
$H_\#^m(Y)$	is the completion of $C_\#^\infty(Y)$ for the norm of $H^m(Y)$; it holds: $D^\alpha u\vert_{y_1=0} = D^\alpha u\vert_{y_1=1}$ and $D^\alpha u\vert_{y_2=0} = D^\alpha u\vert_{y_2=1}$ for $\alpha = (\alpha_1,\alpha_2)$, $\alpha_1,\alpha_2 \geq 0$, $\alpha_1+\alpha_2 \leq m-1$.
$H_\#^m(Y)/R$	is the quotient space.
$Q := (0,T) \times S$	
$\overset{\circ}{C}{}_T^\infty(Q) \subset C^\infty(Q)$	is the subspace of all functions which vanish on ∂S and at $t=T$ along with all derivatives.
$\overset{\circ}{C}{}_{0,T}^\infty(Q;C_\#^\infty(Y))$	is the space of infinitely differentiable functions from Q into $C_\#^\infty(Y)$ which vanish on ∂S, at $t=0$, and $t=T$ along with all derivatives.
$H_T^2(0,T;L_2(S))$	is the subspace of all functions from $H^2(0,T;L_2(S))$ which vanish at $t=T$ along with the first time derivative.
$X_\infty^{211} := L_\infty(0,T;H_0^2(S)) \times L_\infty(0,T;H_0^1(S)) \times L_\infty(0,T;H_0^1(S))$	
$X_\infty^{100} := L_\infty(0,T;H_0^1(S)) \times L_\infty(0,T;L_2(S)) \times L_\infty(0,T;L_2(S))$	
L^ε	is the set of all functions $(\xi^\varepsilon,u_1^\varepsilon,u_2^\varepsilon)$ that are limits of Galerkin approximations in problem (10) w.r.t. weak$*$ topology of X_∞^{211}.

$d_{ij}(\xi,u) := \frac{1}{2}(u_{ix_j} + u_{jx_i} + \xi_{x_i}\xi_{x_j})$ is the in-plane strain tensor. We use notation: $d_{ij} := d_{ij}(\xi,u)$, $d^{\omega}_{ij} := d_{ij}(\xi^{\omega},u^{\omega})$, $\hat{d}_{ij} := d_{ij}(\hat{\xi},\hat{u})$ etc.

$e_{ij}(u) := \frac{1}{2}(u_{ix_j} + u_{jx_i})$ is the linear part of the in-plane strain tensor.

$\|f\| := \|f\|_{L_2(S)}$ is L_2 norm of a function.

$(f,g) := (f,g)_{L_2(S)}$ is L_2 scalar product.

We assume summation over repeating indices. For example,

$$\|\xi_{x_\alpha}\|^2 = \|\xi_{x_\alpha}\| \cdot \|\xi_{x_\alpha}\| := \sum_{\alpha=1}^{2} \|\xi_{x_\alpha}\| \cdot \|\xi_{x_\alpha}\| = \|\xi\|^2_{H^1_0(S)}\,.$$

3. Problem setting

Consider a system of nonlinear equations describing oscillations of a thin plate excited by patches made of a piezoelectric ceramic (see [2]). For simplicity, assume that the plate occupies a rectangular domain S and the piezoelectric patches occupy rectangular domains $S_{P_l} \subset S$ (see Fig. 1). It is assumed that the patches form a periodic structure of the period ε so that the object is completely defined by ε. Denote $S_P := \bigcup_{l=1}^{m} S_{P_l}$ and $S_B := S \setminus S_P$.

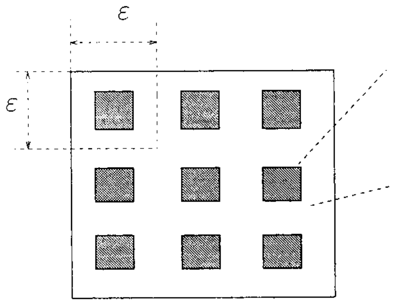
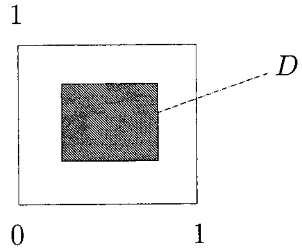


Fig. 1. Plate S with patches S_{P_l}



Unit cell scaled from $\varepsilon \times \varepsilon$ to 1×1

Equations describing the model read

$$\tilde{\rho}\xi_{tt} - \operatorname{div}(\tilde{\mu}\nabla\xi_{tt}) + \Delta(\tilde{\gamma}\,\Delta\xi) - \frac{\partial}{\partial x_\alpha}\,(\tilde{\tau}_{\alpha\beta}(x)\,\xi_{x_\beta}) = Fv_l(t)\,\Delta I_{S_{P_l}} + Gv_l(t)\,\frac{\partial}{\partial x_\alpha}\,(\xi_{x_\alpha} I_{S_{P_l}})\,,$$

$$\tilde{\rho}u_{\alpha tt} - \frac{\partial}{\partial x_\beta}\,\tilde{\tau}_{\alpha\beta}(x) = Gv_l(t)\,\frac{\partial}{\partial x_\alpha}\,I_{S_{P_l}}\,, \quad \text{where} \quad \tilde{\tau}_{\alpha\beta}(x) = \tilde{\ell}_{ij\alpha\beta}(x)\,\frac{1}{2}\,(u_{ix_j} + u_{jx_i} + \xi_{x_i}\xi_{x_j})\,. \tag{1}$$

Here ξ and u_α, $\alpha = 1,2$, are vertical and longitudinal displacements, Δ is the Laplace operator, $v_l(t)$ is the voltage applied to the lth piezoelectric patch S_{P_l} , $I_{S_{P_l}}$ is the indicator function of the lth patch. The indices α and β run over $\{1,2\}$, the index l runs from 1 to m, where m is the number of piezopatches. Summation over repeating indices is assumed. The coefficients F and G are constant; $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\gamma}$, and $\tilde{\ell}_{ij\alpha\beta}$ are discontinuous piecewise constant functions defined as follows:

$$\tilde{\rho}(x) = \begin{cases} \rho_P, & x \in S_P\,, \\ \rho_B, & x \in S_B\,, \end{cases} \qquad \tilde{\mu}(x) = \begin{cases} \mu_P, & x \in S_P\,, \\ \mu_B, & x \in S_B\,, \end{cases}$$

$$\tilde{\gamma}(x) = \begin{cases} \gamma_P, & x \in S_P\,, \\ \gamma_B, & x \in S_B\,. \end{cases} \qquad \tilde{\ell}_{ij\alpha\beta}(x) = \begin{cases} \ell^P_{ij\alpha\beta}, & x \in S_P\,, \\ \ell^B_{ij\alpha\beta}, & x \in S_B\,. \end{cases}$$

Nonzero components of $\ell^P_{ij\alpha\beta}$ and $\ell^B_{ij\alpha\beta}$ are defined as follows:

$$\ell^{\omega}_{1111} = \frac{E_\omega}{1-\sigma^2_\omega}\,, \qquad \ell^{\omega}_{2211} = \frac{E_\omega\sigma_\omega}{1-\sigma^2_\omega}\,, \qquad \ell^{\omega}_{i\neq j\,\alpha\neq\beta} = \frac{E_\omega}{2(1+\sigma_\omega)}\,, \qquad \ell^{\omega}_{1122} = \ell^{\omega}_{2211}\,, \qquad \ell^{\omega}_{2222} = \ell^{\omega}_{1111}\,, \qquad \omega = P, B\,.$$

Here, E_P and E_B are the elastic moduli; σ_P and σ_B are the Poisson ratios. Indices P and B point out to the piezoelectric and base materials, respectively. It is assumed that $v_l(\cdot) \in H^1(0,T)$, $\rho_P > 0$, $\rho_B > 0$, $\mu_P > 0$, $\mu_B > 0$, $\gamma_P > 0$, $\gamma_B > 0$, and

$$\nu d_{\alpha\beta}d_{\alpha\beta} \le \ell^{\omega}_{ij\alpha\beta}d_{ij}d_{\alpha\beta} \le N d_{\alpha\beta}d_{\alpha\beta}\,, \qquad \omega = P, B \tag{2}$$

for any symmetric $d_{ij} \in R^{2\times 2}$, where ν and N are positive constants.

Assume that the controls $v_l(\cdot)$ are being chosen as follows. A distribution $K(t,x) \in H^1(0,T;L_2(S))$ of the voltage over the whole plate S is prescribed and we set

$$v_l(t) = \operatorname{meas}(S_{P_l})^{-1} \iint_{S_{P_l}} K(t,x)\,dx\,. \tag{3}$$

Let

$$K_\varepsilon(t,x) = \begin{cases} K(t,x), & x \in S_B\,, \\ v_l(t), & x \in S_{P_l}\,. \end{cases} \tag{4}$$

It is obvious that $K_\varepsilon \to K$ in $H^1(0,T;L_2(S))$. One can rewrite (1) as follows:

$$\begin{aligned} &\rho\Big(\frac{x}{\varepsilon}\Big)\,\xi_{tt} - \operatorname{div}\Big(\mu\,\Big(\frac{x}{\varepsilon}\Big)\,\nabla\xi_{tt}\Big) + \Delta\Big(\gamma\Big(\frac{x}{\varepsilon}\Big)\,\Delta\xi\Big) - \frac{\partial}{\partial x_\alpha}\,\Big(\tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\xi_{x_\beta}\Big) \\ &= F\Delta\Big(K_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\Big) + G\,\frac{\partial}{\partial x_\alpha}\,\Big(\xi_{x_\alpha}K_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\Big)\,, \\ &\rho\Big(\frac{x}{\varepsilon}\Big)\,u_{\alpha tt} - \frac{\partial}{\partial x_\beta}\,\tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big) = G\,\frac{\partial}{\partial x_\alpha}\,\Big(K_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\Big)\,, \qquad \alpha = 1,2\,, \end{aligned} \tag{5}$$

where

$$\tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big) = \ell_{ij\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\frac{1}{2}\,(u_{ix_j} + u_{jx_i} + \xi_{x_i}\xi_{x_j})\,.$$

Here ρ, μ, γ, I, $\ell_{ij\alpha\beta}$ are Y periodic functions defined on Y as follows:

$$\begin{aligned} &I(y) = I_D(y)\,, \qquad \rho(y) = I_D(y)\,\rho_P + (1 - I_D(y))\,\rho_B\,, \qquad \mu(y) = I_D(y)\,\mu_P + (1 - I_D(y))\,\mu_B\,, \\ &\gamma(y) = I_D(y)\,\gamma_P + (1 - I_D(y))\,\gamma_B\,, \qquad \ell_{ij\alpha\beta}(y) = I_D(y)\,\ell^P_{ij\alpha\beta} + (1 - I_D(y))\,\ell^B_{ij\alpha\beta}\,, \end{aligned} \tag{6}$$

where I_D is the indicator function of D (see Fig. 1 right). It follows from (2) that

$$\nu d_{\alpha\beta} d_{\alpha\beta} \le \ell_{ij\alpha\beta}(y)\,d_{ij}d_{\alpha\beta} \le N d_{\alpha\beta}d_{\alpha\beta} \tag{7}$$

for any $y \in Y$ and any symmetric $d_{ij} \in R^{2\times 2}$.

Boundary and initial conditions are

$$\begin{aligned} &\xi|_{\partial S} = 0\,, \qquad \partial\xi/\partial\vec{n}|_{\partial S} = 0\,, \qquad \xi|_{t=0} = \xi_0\,, \qquad \xi_t|_{t=0} = \xi_0'\,, \\ &u_\alpha|_{\partial S} = 0\,, \qquad u_\alpha|_{t=0} = u_{\alpha 0}\,, \qquad u_{\alpha t}|_{t=0} = u_{\alpha 0}'\,, \qquad \alpha = 1,2\,. \end{aligned} \tag{8}$$

Note that (5) should be supplied with the following interface conditions:

$$[\xi] = 0\,, \qquad \Big[\frac{\partial\xi}{\partial\vec{n}}\Big] = 0\,, \qquad [\gamma\,\Delta\xi] = 0\,, \qquad \Big[\frac{\partial}{\partial\vec{n}}\,\gamma\,\Delta\xi\Big] = 0\,, \qquad [u_\alpha] = 0\,, \qquad [\tau_{\alpha\beta}\cdot n_\beta] = 0\,, \qquad \alpha = 1,2\,, \tag{9}$$

that hold on the boundary between S_P and S_B because of the integration by parts when deriving (5) from a weak formulation. Here, $[\cdot]$ denotes the jump of a function on the boundary between S_P and S_B. We do not pay any attention to these conditions because we will go back to the weak formulation.

Definition 1: We say that functions

$$\xi^\varepsilon \in L_2(0,T;H_0^2(S))\,, \qquad u_\alpha^\varepsilon \in L_2(0,T;H_0^1(S))\,, \qquad \alpha = 1,2\,,$$

form a weak solution to system (5)–(9) if the following equality holds:

$$\begin{aligned} &\int_0^T\!\!\iint_S \Big[\rho\Big(\frac{x}{\varepsilon}\Big)\,\xi^\varepsilon\varphi_{tt} + \mu\Big(\frac{x}{\varepsilon}\Big)\,\nabla\xi^\varepsilon\nabla\varphi_{tt} + \gamma\Big(\frac{x}{\varepsilon}\Big)\,\Delta\xi^\varepsilon\Delta\varphi + \tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\xi^\varepsilon_{x_\beta}\varphi_{x_\alpha} - FK_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\,\Delta\varphi \\ &\qquad + GK_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\,\xi^\varepsilon_{x_\alpha}\varphi_{x_\alpha} + \rho\Big(\frac{x}{\varepsilon}\Big)\,u^\varepsilon_\alpha\psi_{\alpha tt} + \tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\psi_{\alpha x_\beta} + GK_\varepsilon(t,x)\,I\Big(\frac{x}{\varepsilon}\Big)\,\psi_{\alpha x_\alpha}\Big]\,dx\,dt \\ &+ \iint_S \Big[\rho\Big(\frac{x}{\varepsilon}\Big)\,(\xi_0\varphi_t(0,x) - \xi_0'\varphi(0,x) + u_{\alpha 0}\psi_{\alpha t}(0,x) - u_{\alpha 0}'\psi_\alpha(0,x)) \\ &\qquad + \mu\Big(\frac{x}{\varepsilon}\Big)\,(\nabla\xi_0\nabla\varphi_t(0,x) - \nabla\xi_0'\nabla\varphi(0,x))\Big]\,dx \;=\; 0 \end{aligned} \tag{10}$$

for all

$$\varphi \in H_T^2(0,T;H_0^1(S)) \cap L_2(0,T;H_0^2(S))\,, \qquad \psi_\alpha \in H_T^2(0,T;L_2(S)) \cap L_2(0,T;H_0^1(S))\,.$$

Proposition 1: *Let $\xi^{\varepsilon m}$ and $u_\alpha^{\varepsilon m}$ be Galerkin approximations computed using the above definition of solutions. Then:*

a) If $\xi_0 \in H_0^2(S)$, $\xi_0' \in H_0^1(S)$, $u_{\alpha 0} \in H_0^1(S)$, $u_{\alpha 0}' \in L_2(S)$, and $K \in H^1(0,T;L_2(S))$, then there exists a constant C independent of ε, m such that

$$\|\xi^{\varepsilon m}\|^2_{H_0^2(S)} + \|\xi_t^{\varepsilon m}\|^2_{H_0^1(S)} + \|u_\alpha^{\varepsilon m}\|^2_{H_0^1(S)} + \|u_{\alpha t}^{\varepsilon m}\|^2_{L_2(S)} \le C \tag{11}$$

for any $t \in [0,T]$.

b) *If* $\xi_0 = 0$, $\xi_0' = 0$, $u_{\alpha 0} = 0$, $u_{\alpha 0}' = 0$, $K \in H^2(0,T;L_2(S))$, *and* $K|_{t=0} = 0$, *then there exists a constant* C *independent of* ε, m *such that*

$$\|\xi_t^{\varepsilon m}\|^2_{H_0^2(S)} + \|\xi_{tt}^{\varepsilon m}\|^2_{H_0^1(S)} + \|u_{\alpha t}^{\varepsilon m}\|^2_{H_0^1(S)} + \|u_{\alpha tt}^{\varepsilon m}\|^2_{L_2(S)} \le C \tag{12}$$

for any $t \in [0,T]$.

Remark 1: The proof of the second part of the proposition is based on the formal time differentiation of equations defining $\xi^{\varepsilon m}$ and $u_\alpha^{\varepsilon m}$. The term $K_{\varepsilon t}$ (see (3) and (4)) arises on the right-hand-side of these equations. The assumption $K \in H^2(0,T;L_2(S))$ is necessary to estimate terms containing $K_{\varepsilon t}$. The requirements $\xi_0 = 0$, $\xi_0' = 0$, $u_{\alpha 0} = 0$, $u_{\alpha 0}' = 0$, and $K|_{t=0} = 0$ provide the so-called compatibility conditions (see e.g. [9]), which guarantees that the formally differentiated equations define time derivatives of $\xi^{\varepsilon m}$ and $u_\alpha^{\varepsilon m}$. For example, one of the compatibility conditions reads $\gamma(x/\varepsilon)\,\Delta\xi_0 \in H_0^1(S)$. This can be satisfied by very special choice of ξ_0 using the structure of the discontinuous function $\gamma(x/\varepsilon)$. To avoid such difficulties, we set $\xi_0 = 0$. Similar arguments explain the choice of the other initial conditions to be homogeneous.

Proof: The proof of the first part of the proposition can be found in [2]. Prove the second part. Let

$$\xi^{\varepsilon m} = \sum_{i=0}^{m} a_i^m(t)\,\omega_i\,, \qquad u_\alpha^{\varepsilon m} = \sum_{i=0}^{m} b_{\alpha i}^m(t)\,\eta_i\,, \qquad \alpha = 1,2\,,$$

where $\{\omega_i\}_{i=1}^\infty$, $\{\eta_i\}_{i=1}^\infty$ are bases of $H_0^2(S)$ and $H_0^1(S)$, respectively, and the coefficients $a_i^m(t)$, $b_{\alpha i}^m(t)$ satisfying the initial conditions $a_i^m(0) = 0$, $d/dt\, a_i^m(0) = 0$, $b_{\alpha i}^m(0) = 0$, $d/dt\, b_{\alpha i}^m(0) = 0$ are found from (10), that is

$$\iint_S \left(\rho\Big(\frac{x}{\varepsilon}\Big)\,\xi_{tt}^{\varepsilon m}\varphi + \mu\Big(\frac{x}{\varepsilon}\Big)\,\nabla\xi_{tt}^{\varepsilon m}\nabla\varphi + \gamma\Big(\frac{x}{\varepsilon}\Big)\,\Delta\xi^{\varepsilon m}\Delta\varphi + \tau_{\alpha\beta}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big)\,\xi_{x_\beta}^{\varepsilon m}\varphi_{x_\alpha}\right) dx$$

$$= \iint_{S_P} FK_\varepsilon(t,x)\,\Delta\varphi\,dx - \iint_{S_P} GK_\varepsilon(t,x)\,\xi_{x_\alpha}^{\varepsilon m}\varphi_{x_\alpha}\,dx\,,$$

$$\iint_S \left(\rho\Big(\frac{x}{\varepsilon}\Big)\,u_{\alpha tt}^{\varepsilon m}\psi_\alpha + \tau_{\alpha\beta}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big)\,\psi_{\alpha x_\beta}\right) dx = -\iint_{S_P} GK_\varepsilon(t,x)\,\psi_{\alpha x_\alpha}\,dx$$

for all $t \in [0,T]$, $\varphi \in \operatorname{span}\{\omega_1,\ldots,\omega_m\}$, $\psi_\alpha \in \operatorname{span}\{\eta_1,\ldots,\eta_m\}$. Here

$$\tau_{ij}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big) = \ell_{ij\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\frac{1}{2}\left[\frac{\partial u_i^{\varepsilon m}}{\partial x_j} + \frac{\partial u_j^{\varepsilon m}}{\partial x_i} + \frac{\partial\xi^{\varepsilon m}}{\partial x_i}\,\frac{\partial\xi^{\varepsilon m}}{\partial x_j}\right], \qquad i,j = 1,2\,.$$

Formal differentiation w.r.t. t yields

$$\iint_S \left(\rho\Big(\frac{x}{\varepsilon}\Big)\,w_{tt}^{\varepsilon m}\varphi + \mu\Big(\frac{x}{\varepsilon}\Big)\,\nabla w_{tt}^{\varepsilon m}\nabla\varphi + \gamma\Big(\frac{x}{\varepsilon}\Big)\,\Delta w^{\varepsilon m}\Delta\varphi + \dot\tau_{\alpha\beta}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big)\,\xi_{x_\beta}^{\varepsilon m}\varphi_{x_\alpha} + \tau_{\alpha\beta}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big)\,w_{x_\beta}^{\varepsilon m}\varphi_{x_\alpha}\right) dx$$

$$= \iint_{S_P} FK_{\varepsilon t}(t,x)\,\Delta\varphi\,dx - \iint_{S_P} GK_{\varepsilon t}(t,x)\,\xi_{x_\alpha}^{\varepsilon m}\varphi_{x_\alpha}\,dx - \iint_{S_P} GK_\varepsilon(t,x)\,w_{x_\alpha}^{\varepsilon m}\varphi_{x_\alpha}\,dx\,,$$

$$\iint_S \left(\rho\Big(\frac{x}{\varepsilon}\Big)\,v_{\alpha tt}^{\varepsilon m}\psi_\alpha + \dot\tau_{\alpha\beta}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big)\,\psi_{\alpha x_\beta}\right) dx = -\iint_{S_P} GK_{\varepsilon t}(t,x)\,\psi_{\alpha x_\alpha}\,dx\,.$$

Here

$$\dot\tau_{ij}^{\varepsilon m}\Big(\frac{x}{\varepsilon}\Big) = \ell_{ij\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\,\frac{1}{2}\left[\left(\frac{\partial v_i^{\varepsilon m}}{\partial x_j} + \frac{\partial v_j^{\varepsilon m}}{\partial x_i}\right) + \frac{\partial w^{\varepsilon m}}{\partial x_i}\,\frac{\partial\xi^{\varepsilon m}}{\partial x_j} + \frac{\partial w^{\varepsilon m}}{\partial x_j}\,\frac{\partial\xi^{\varepsilon m}}{\partial x_i}\right], \qquad i,j = 1,2\,.$$

Since the compatibility conditions are fulfilled, it holds $w^{\varepsilon m} = \xi_t^{\varepsilon m}$ and $v_\alpha^{\varepsilon m} = u_{\alpha t}^{\varepsilon m}$.

We set $\varphi = w_t$, $\psi_\alpha = v_{\alpha t}$ and sum up the equations. This yields the following equality:

$$\frac{1}{2}\,\frac{d}{dt}\left(\|\rho w_t^{\varepsilon m}\|^2 + \|\mu\nabla w_t^{\varepsilon m}\|^2 + \|\gamma\,\Delta w^{\varepsilon m}\|^2 + \|\rho v_{\alpha t}^{\varepsilon m}\|^2 + \iint_S \ell_{\alpha\beta ij}^{-1}\dot\tau_{\alpha\beta}^{\varepsilon m}\dot\tau_{ij}^{\varepsilon m}\,dx + \iint_S \tau_{\alpha\beta}^{\varepsilon m}w_{x_\alpha}^{\varepsilon m}w_{x_\beta}^{\varepsilon m}\,dx\right)$$

$$= \frac{3}{2}\iint_S \dot\tau_{\alpha\beta}^{\varepsilon m}w_{x_\alpha}^{\varepsilon m}w_{x_\beta}^{\varepsilon m}\,dx + \iint_{S_P} FK_{\varepsilon t}(t,x)\,\frac{d}{dt}\,\Delta w^{\varepsilon m}\,dx - \iint_{S_P} GK_{\varepsilon t}(t,x)\,\frac{d}{dt}\,(\ell_{\alpha\alpha ij}^{-1}\dot\tau_{ij}^{\varepsilon m})\,dx$$

$$+ \iint_{S_P} GK_{\varepsilon t}w_{x_\alpha}^{\varepsilon m}w_{x_\alpha}^{\varepsilon m}\,dx - \iint_{S_P} GK_\varepsilon(t,x)\,w_{x_\alpha}^{\varepsilon m}w_{x_\alpha}^{\varepsilon m}\,dx\,, \tag{13}$$

where $\ell^{-1}_{\alpha\beta ij}$ is the inverse of $\ell_{\alpha\beta ij}$, that is

$$\ell^{-1}_{\alpha\beta pq}\ell_{ijpq}s_{ij} = s_{\alpha\beta} \tag{14}$$

for any symmetric $s \in R^{2\times 2}$. Note that

$$\nu/N\dot{\tau}^{\varepsilon m}_{\alpha\beta}\dot{\tau}^{\varepsilon m}_{\alpha\beta} \;\le\; \ell^{-1}_{\alpha\beta ij}\dot{\tau}^{\varepsilon m}_{\alpha\beta}\dot{\tau}^{\varepsilon m}_{ij} \;\le\; 1/\nu\,\dot{\tau}^{\varepsilon m}_{\alpha\beta}\dot{\tau}^{\varepsilon m}_{\alpha\beta} \tag{15}$$

because of (2) and (14).

Integration of (13) over t gives

$$\begin{aligned}
&\|\rho w^{\varepsilon m}_t\|^2 + \|\mu\nabla w^{\varepsilon m}_t\|^2 + \|\gamma\,\Delta w^{\varepsilon m}\|^2 + \|\rho v^{\varepsilon m}_{\alpha t}\|^2 + \iint_S \ell^{-1}_{\alpha\beta ij}\dot{\tau}^{\varepsilon m}_{\alpha\beta}\dot{\tau}^{\varepsilon m}_{ij}\,dx\\
&= -\iint_S \tau^{\varepsilon m}_{\alpha\beta}w^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\beta}\,dx + 3\int_0^t\iint_S \dot{\tau}^{\varepsilon m}_{\alpha\beta}w^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\beta}\,dx + 2\iint_{S_P} FK_{\varepsilon t}(t,x)\,\Delta w^{\varepsilon m}\,dx\\
&\quad -2\int_0^t\iint_{S_P} FK_{\varepsilon tt}(t,x)\,\Delta w^{\varepsilon m}\,dx - 2\iint_{S_P} GK_{\varepsilon t}(t,x)\,\ell^{-1}_{\alpha\alpha ij}\dot{\tau}^{\varepsilon m}_{ij}\,dx + 2\int_0^t\iint_{S_P} GK_{\varepsilon tt}(t,x)\,\ell^{-1}_{\alpha\alpha ij}\dot{\tau}^{\varepsilon m}_{ij}\,dx\\
&\quad +2\iint_{S_P} GK_{\varepsilon t}w^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\alpha}\,dx - 2\iint_{S_P} GK_{\varepsilon}(t,x)\,w^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\alpha}\,dx\,.
\end{aligned} \tag{16}$$

Using the inequalities of Schwartz and Gagliardo-Nirenberg yields

$$\|w^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\beta}\| \le \|\nabla w^{\varepsilon m}\|^2_{L_4(S)} \le C\|\nabla w^{\varepsilon m}\|\,\|w^{\varepsilon m}\|_{H^2_0(S)} \le C\|w^{\varepsilon m}\|_{H^2_0(S)}\,.$$

The final inequality is true because of the boundness of $\|\nabla w^{\varepsilon m}\|$ due to (11). Using that $\|\tau^{\varepsilon m}_{\alpha\beta}\|^2 \le C$ due to (11), we obtain from (16)

$$\|w^{\varepsilon m}\|^2_{H^2_0(S)} + \|w^{\varepsilon m}_t\|^2_{H^1_0(S)} + \|v^{\varepsilon m}_{\alpha t}\|^2 + \|\dot{\tau}^{\varepsilon m}_{\alpha\beta}\|^2 \le C\,. \tag{17}$$

Taking into account that

$$\frac{1}{2}\left(\frac{\partial v^{\varepsilon m}_\alpha}{\partial x_\beta} + \frac{\partial v^{\varepsilon m}_\beta}{\partial x_\alpha}\right) = \ell^{-1}_{\alpha\beta ij}\dot{\tau}^{\varepsilon m}_{ij} - \frac{1}{2}\left(\xi^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\beta} + w^{\varepsilon m}_{x_\alpha}\xi^{\varepsilon m}_{x_\beta}\right),$$

we obtain

$$\begin{aligned}
\left\|\frac{1}{2}\left(\frac{\partial v^{\varepsilon m}_\alpha}{\partial x_\beta} + \frac{\partial v^{\varepsilon m}_\beta}{\partial x_\alpha}\right)\right\| &\le C(\|\dot{\tau}^{\varepsilon m}_{\alpha\beta}\| + \|\xi^{\varepsilon m}_{x_\alpha}w^{\varepsilon m}_{x_\beta}\|) \le C(\|\dot{\tau}^{\varepsilon m}_{\alpha\beta}\| + \|\xi^{\varepsilon m}_{x_\alpha}\|^2_{L_4(S)} + \|w^{\varepsilon m}_{x_\alpha}\|^2_{L_4(S)})\\
&\le C(\|\dot{\tau}^{\varepsilon m}_{\alpha\beta}\| + \|\xi^{\varepsilon m}\|^2_{H^2_0(S)} + \|w^{\varepsilon m}\|^2_{H^2_0(S)}) \le C
\end{aligned}$$

using (11), (17) and the continuous embedding $H^2_0(S) \subset W^1_4(S)$. The inequality of Korn implies

$$\|v^{\varepsilon m}_\alpha\|^2_{H^1_0(S)} \le C\,,$$

which completes the proof of proposition 1. □

Proposition 2: *Let L^ε be the set of all limit points (in weak$*$ topology of X^{211}_∞) of all possible Galerkin approximations in* (10). *Then:*

a) *$L^\varepsilon \ne \emptyset$ and each $(\xi^\varepsilon, u^\varepsilon_1, u^\varepsilon_2) \in L^\varepsilon$ is a weak solution of* (10).

b) *If the first assumption of proposition 1 holds, then there exists a constant C independent of ε such that*

$$\|\xi^\varepsilon\|^2_{L_\infty(0,T;\,H^2_0(S))} + \|\xi^\varepsilon_t\|^2_{L_\infty(0,T;\,H^1_0(S))} + \|u^\varepsilon_\alpha\|^2_{L_\infty(0,T;\,H^1_0(S))} + \|u^\varepsilon_{\alpha t}\|^2_{L_\infty(0,T;\,L_2(S))} \le C \tag{18}$$

for any $(\xi^\varepsilon, u^\varepsilon_1, u^\varepsilon_2) \in L^\varepsilon$.

c) *If the second assumption of proposition 1 holds, then there exists a constant C independent of ε such that*

$$\|\xi^\varepsilon_t\|^2_{L_\infty(0,T;\,H^2_0(S))} + \|\xi^\varepsilon_{tt}\|^2_{L_\infty(0,T;\,H^1_0(S))} + \|u^\varepsilon_{\alpha t}\|^2_{L_\infty(0,T;\,H^1_0(S))} + \|u^\varepsilon_{\alpha tt}\|^2_{L_\infty(0,T;\,L_2(S))} \le C \tag{19}$$

for any $(\xi^\varepsilon, u^\varepsilon_1, u^\varepsilon_2) \in L^\varepsilon$.

Proof: Nonemptiness of L^ε follows from (11). The proof that each $(\xi^\varepsilon, u^\varepsilon_1, u^\varepsilon_2) \in L^\varepsilon$ is a weak solution of (10) can be found in [2]. It follows from (11) and (12) that all necessary time derivatives in (18) and (19) exist, and that (18) and (19) hold. □

4. Homogenization

The following holds for any $(\xi^\varepsilon, u_1^\varepsilon, u_2^\varepsilon) \in L^\varepsilon$ due to Proposition 2:

$$\|\xi^\varepsilon\|^2_{L_\infty(0,T;H_0^2(S))} + \|u_1^\varepsilon\|^2_{L_\infty(0,T;H_0^1(S))} + \|u_2^\varepsilon\|^2_{L_\infty(0,T;H_0^1(S))} \le C\,,$$

with C independent from ε. Therefore, the sequence $(\xi^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ contains a weak$*$ converging subsequence in X_∞^{211}. Now we derive equations that define limit functions of such subsequences (effective equations). We will show that these equations have a unique solution under assumptions of the second part of Proposition 1, and that the coefficients of the effective equations are independent of the choice of subsequences. This yields that the sequence $(\xi^\varepsilon, u_1^\varepsilon, u_2^\varepsilon) \in L^\varepsilon$ converges weak$*$ in X_∞^{211} to the solution of the effective system. Similar arguments show that $(\xi_t^\varepsilon, u_{1\,t}^\varepsilon, u_{2\,t}^\varepsilon)$ converges weak$*$ in X_∞^{211}, and $(\xi_{tt}^\varepsilon, u_{1\,tt}^\varepsilon, u_{2\,tt}^\varepsilon)$ converges weak$*$ in X_∞^{100} to time derivatives of the solution of the effective system. Then, using Corollary 4 of Simon [10] and Theorem 16.1 of Lions [11], we conclude that $(\xi^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ and $(\xi_t^\varepsilon, u_{1t}^\varepsilon, u_{2t}^\varepsilon)$ converges strongly in $C([0,T];\,H_0^{2-s}(S)) \times C([0,T];\,H_0^{1-s}(S)) \times C([0,T];\,H_0^{1-s}(S))$ for any positive real s. In particular, ξ^ε and ξ_t^ε converge uniformly on $[0,T]\times \bar S$.

Now we apply two-scale convergence to derivation of the effective equations. For the pioneering works on two-scale convergence for time-independent problems we refer to [3] and [4]. Two-scale convergence for time-dependent problems was considered in [5]. These results were generalized in [6].

Let us reproduce the definition of two-scale convergence of functions depending on additional parameters (Definition 6.8 of Haller [6]): Let $v_\varepsilon \in L_2(Q)$, $v_0 \in L_2(Q\times Y)$. It is said that $v_\varepsilon \overset{\text{2-scale}}{\longrightarrow} v_0$, if

$$\lim_{\varepsilon\to 0} \int_0^T \iint_S v_\varepsilon(t,x)\, \psi(t,x,x/\varepsilon)\, dx\, dt = \int_0^T \iint_S \iint_Y v_0(t,x,y)\, \psi(t,x,y)\, dy\, dx\, dt$$

for all $\psi \in \overset{\circ}{C}{}^\infty_{0,T}(Q; C_\#^\infty(Y))$.

It is proved (Theorem 6.15 of Haller [6]) that all properties of two-scale convergence hold, if the test functions in the above definition are replaced by more general test functions of the form: $\psi(t,x,y) = f(t,x)\, g(y)\, \sigma(t,x,y)$, where $f \in L_\infty(Q)$, $g \in L_{\infty\#}(Y)$, and $\sigma \in C^\infty(Q; C_\#^\infty(Y))$ (not necessary vanishes). Theorem 6.12 of Haller [6] is a generalization of results of [4, 5] about two-scale convergence of bounded functional sequences. Let $\|v_\varepsilon\|_{L_2(0,T;\,H^m(S))} \le C$ with C independent of ε and $m = 1,2$. Then there exist ε_j, $v(t,x) \in L_2(0,T;H^m(S))$, and $\bar v(t,x,y) \in L_2(Q; H_\#^m(Y))$ such that

$$v^{\varepsilon_j} \overset{\text{weak}*}{\longrightarrow} v \quad \text{in} \quad L_\infty(0,T;H^m(S))\,; \qquad D^k v^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} D^k v\,, \qquad k\in \overline{0,m-1}\,; \qquad D^m v^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} D^m v + D_y^m \bar v\,.$$

Due to the above result, there exist ε_j, $\xi(t,x) \in L_2(0,T;H_0^2(S))$, $u_\alpha(t,x) \in L_2(0,T;H_0^1(S))$, $\bar\xi(t,x,y) \in L_2(Q;H_\#^2(Y))$, and $\bar u_\alpha(t,x,y) \in L_2(Q;H_\#^1(Y))$ such that

$$\xi^{\varepsilon_j} \overset{\text{weak}*}{\longrightarrow} \xi \quad \text{in} \quad L_\infty(0,T;H_0^2(S))\,, \qquad \xi^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} \xi\,, \qquad \nabla\xi^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} \nabla\xi\,, \qquad \Delta\xi^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} \Delta\xi + \Delta_y\bar\xi\,,$$

$$u_\alpha^{\varepsilon_j} \overset{\text{weak}*}{\longrightarrow} u_\alpha \quad \text{in} \quad L_\infty(0,T;H_0^1(S))\,, \qquad u_\alpha^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} u_\alpha\,, \qquad u_{\alpha x_\beta}^{\varepsilon_j} \overset{\text{2-scale}}{\longrightarrow} u_{\alpha x_\beta} + \bar u_{\alpha y_\beta}\,.$$

Moreover, from (18) and from the compact embedding $H_0^2(S) \subset W_q^1(S)$, for any $q > 1$, we conclude that $\{\xi^{\varepsilon_j}\}$ is relative compact in $C((0,T);W_q^1(S))$ for any $q>1$ (see [10]). So, we can assume that $\xi^{\varepsilon_j} \to \xi$ in $C\big([0,T];W_q^1(S)\big)$ for any $q>1$.

To obtain effective equations defining ξ and u_α, set

$$\varphi(t,x) = \eta(t,x) + \varepsilon^2\phi(t,x,x/\varepsilon)\,, \qquad \psi_\alpha(t,x) = \chi_\alpha(t,x) + \varepsilon\theta_\alpha(t,x,x/\varepsilon)\,,$$

where $\eta(t,x)$, $\chi_\alpha(t,x) \in \overset{\circ}{C}{}^\infty_T(Q)$, and $\phi(t,x,y)$, $\theta_\alpha(t,x,y) \in \overset{\circ}{C}{}^\infty_{0,T}(Q; C_\#^\infty(Y))$. Substituting these functions into (10) yields (we omit the index j for brevity)

$$\begin{aligned}
&\int_0^T\iint_S \Big\{\rho\Big(\frac{x}{\varepsilon}\Big)\, \xi^\varepsilon[\eta_{tt} + \varepsilon^2\phi_{tt}] + \mu\Big(\frac{x}{\varepsilon}\Big)\, \nabla\xi^\varepsilon[\nabla\eta_{tt} + \varepsilon^2(\ldots)] + \gamma\Big(\frac{x}{\varepsilon}\Big)\, \Delta\xi^\varepsilon[\Delta\eta + \Delta_y\phi + \varepsilon^2(\ldots)]\\
&\qquad + \tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\, \xi^\varepsilon_{x_\beta}[\eta_{x_\alpha} + \varepsilon^2(\ldots)] - FK_\varepsilon(t,x)\, I\Big(\frac{x}{\varepsilon}\Big)\, [\Delta\eta + \Delta_y\phi + \varepsilon^2(\ldots)] + GK_\varepsilon(t,x)\, I\Big(\frac{x}{\varepsilon}\Big)\, \xi^\varepsilon_{x_\alpha}[\eta_{x_\alpha} + \varepsilon^2(\ldots)]\\
&\qquad + \rho\Big(\frac{x}{\varepsilon}\Big)\, u_\alpha^\varepsilon[\chi_{\alpha tt} + \varepsilon\theta_{\alpha tt}] + \tau_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big)\, [\chi_{\alpha x_\beta} + \theta_{\alpha y_\beta} + \varepsilon(\ldots)] + GK_\varepsilon(t,x)\, I\Big(\frac{x}{\varepsilon}\Big)\, [\chi_{\alpha x_\alpha} + \theta_{\alpha y_\alpha} + \varepsilon(\ldots)]\Big\}\, dx\, dt\\
&+ \iint_S \Big\{\rho\Big(\frac{x}{\varepsilon}\Big)\, [\xi_0\eta_t(0,x) - \xi_0'\eta(0,x) + u_{\alpha 0}\chi_{\alpha t}(0,x) - u_{\alpha 0}'\chi_\alpha(0,x)]\\
&\qquad \times \mu\Big(\frac{x}{\varepsilon}\Big)\, [\nabla\xi_0\nabla\eta_t(0,x) - \nabla\xi_0'\nabla\eta(0,x)]\Big\}\, dx = 0\,.
\end{aligned} \tag{20}$$

The symbol $(\ldots)$ denotes terms with the multipliers ε^{-1} and ε^0, whereas the symbol $(\dot{\cdot}.)$ denotes terms with the multipliers ε^0. These terms appear when applying differential operators of (10) to the functions $\phi(t,x,x/\varepsilon)$ and $\theta_\alpha(t,x,x/\varepsilon)$. For example, the term $(\ldots)$ in the first line of (20) is equal to $\nabla_\times\phi_{tt} + (1/\varepsilon)\,\nabla_y\phi_{tt}$, where $\nabla_\times$ and ∇_y denote the gradients with respect to the second and the third variables of $\phi(t,x,x/\varepsilon)$. Considering

$$\begin{aligned}
&\rho(y)\;[\eta_{tt}(t,x)+\varepsilon^2\phi_{tt}(t,x,y)]\,, && \mu(y)\;[\nabla\eta_{tt}(t,x)+\varepsilon^2(\ldots)]\,,\\
&\gamma(y)\;[\Delta\eta(t,x)+\Delta_y\phi(t,x,y)+\varepsilon^2(\ldots)]\,, && \ell_{ija\beta}(y)\;\xi_{x_\beta}(t,x)\;[\eta_{x_a}(t,x)+\varepsilon^2(\ldots)]\,,\\
&I(y)\;[\Delta\eta(t,x)+\Delta_y\phi(t,x,y)+\varepsilon^2(\ldots)]\,, && I(y)\;\xi_{x_a}(t,x)\;[\eta_{x_a}(t,x)+\varepsilon^2(\ldots)]\,,\\
&\rho(y)\;[\chi_{tt}(t,x)+\varepsilon\theta_{tt}(t,x,y)]\,, && \ell_{ija\beta}(y)\;[\chi_{\alpha x_\beta}(t,x)+\theta_{\alpha y_\beta}(t,x,y)+\varepsilon(\dot{\cdot}.)]\,,\\
&I(y)\;[\chi_{\alpha x_a}(t,x)+\theta_{\alpha y_a}(t,x,y)+\varepsilon(\dot{\cdot}.)]
\end{aligned}$$

as test functions, one can pass to the two-scale limit in (20) taking into account that $\varepsilon^2(\ldots)$ and $\varepsilon(\dot{\cdot}.)$ converge uniformly to 0, $K_\varepsilon \to K$ in $H^1(0,T;L_2(S))$, and $\xi^\varepsilon_{x_\alpha} \to \xi_{x_\alpha}$ in $C([0,T];L_q(S))$ for any $q>1$. Since the last system contains two test functions, the limiting system will be split into two ones.

The first, effective, system defining the limit functions ξ and u_α reads

$$\begin{aligned}
&\int_0^T\iint_S\iint_Y \{\rho(y)\;\xi\eta_{tt}+\mu(y)\;\nabla\xi\nabla\eta_{tt}+\gamma(y)\;[\Delta\xi+\Delta_y\bar{\xi}]\;\Delta\eta\\
&\qquad+\ell_{ija\beta}(y)\,\tfrac{1}{2}(u_{ix_j}+u_{jx_i}+\xi_{x_i}\xi_{x_j}+\bar{u}_{iy_j}+\bar{u}_{jy_i})\;\xi_{x_\beta}\eta_{x_a}-FK(t,x)\;I(y)\;\Delta\eta+GK(t,x)\;I(y)\;\xi_{x_a}\eta_{x_a}\\
&\qquad+\rho(y)\;u_\alpha\chi_{att}+\ell_{ija\beta}(y)\,\tfrac{1}{2}(u_{ix_j}+u_{ix_j}+\xi_{x_i}\xi_{x_j}+\bar{u}_{iy_j}+\bar{u}_{jy_i})\;\chi_{\alpha x_\beta}+GK(t,x)\;I(y)\;\chi_{\alpha x_a}\}\;dy\;dx\;dt\\
&+\iint_S\iint_Y\{\rho(y)\;[\xi_0\eta_t(0,x)-\xi_0'\eta(0,x)+u_{\alpha0}\chi_{\alpha t}(0,x)-u_{\alpha0}'\chi_\alpha(0,x)]\\
&\qquad+\mu(y)\;[\nabla\xi_0\nabla\eta_t(0,x)-\nabla\xi_0'\nabla\eta(0,x)]\}\;dy\;dx=0\,.
\end{aligned}\tag{21}$$

The so-called cell equations defining auxiliary functions $\bar{\xi}$ and $\bar{u}_\alpha$ read

$$\begin{aligned}
&\int_0^t\iint_S\iint_Y\{\gamma(y)\;[\Delta\xi+\Delta_y\bar{\xi}]\;\Delta_y\phi-FK(t,x)\;I(y)\;\Delta_y\phi\\
&\qquad+\ell_{ija\beta}(y)\,\tfrac{1}{2}(u_{ix_j}+u_{jx_i}+\xi_{x_i}\xi_{x_j}+\bar{u}_{iy_j}+\bar{u}_{jy_i})\;\theta_{\alpha y_\beta}+GK(t,x)\;I(y)\;\theta_{\alpha x_a}\}\;dy\;dx\;dt=0\,.
\end{aligned}\tag{22}$$

Here y is the independent variable, whereas x is treated as a parameter. Since the system (22) is linear, the superposition principle yields

$$\bar{\xi}(t,x,y)=N(y)\;\Delta\xi+M(y)\;FK(t,x)\,,\tag{23}$$

$$\bar{u}_i(t,x,y)=N_{imn}(y)\,\tfrac{1}{2}(u_{mx_n}+u_{nx_m}+\xi_{x_n}\xi_{x_m})+M_i(y)\;GK(t,x)\,,\tag{24}$$

where $N,M\in H^2_\#(Y)$ and $N_{imn},M_i\in H^1_\#(Y)$ are unknown functions. Substituting them in (22) and some computation yield

$$\int_0^t\iint_S\iint_Y\{\Delta\xi\gamma(y)\;[1+\Delta_yN]+FK(t,x)\;[\gamma(y)\;\Delta_yM-I(y)]\}\;\Delta_y\phi\;dy\;dx\;dt=0\,,$$

$$\begin{aligned}
&\int_0^T\iint_S\iint_Y\left\{d_{mn}(\xi,u)\;\ell_{ija\beta}(y)\;\left[\delta_{im}\delta_{jn}+\frac{1}{2}\left(\frac{\partial N_{imn}}{\partial y_j}+\frac{\partial N_{jmn}}{\partial y_i}\right)\right]\right.\\
&\qquad\left.+GK(t,x)\;\left[\ell_{ija\beta}(y)\frac{1}{2}\left(\frac{\partial M_i}{\partial y_j}+\frac{\partial M_j}{\partial y_i}\right)+\delta_{\alpha\beta}I(y)\right]\right\}\;\theta_{\alpha y_\beta}\;dy\;dx\;dt=0\,.
\end{aligned}$$

Because of the superposition principle one can seek N, M, N_{imn}, and M_i separately. Taking the test functions of the form $\phi(t,x,y)=\phi^1(t,x)\;\phi^2(y)$, $\theta_\alpha(t,x,y)=\theta^1_\alpha(t,x)\;\theta^2_\alpha(y)$, we obtain

$$\iint_Y\gamma(y)\;(1+\Delta_yN)\;\Delta_y\phi^2\;dy=0\,,\tag{25}$$

$$\iint_Y(\gamma(y)\;\Delta_yM-I(y))\;\Delta_y\phi^2\;dy=0\,,\tag{26}$$

$$\begin{aligned}
&\iint\limits_Y \ell_{ij1\beta}(y)\left[2\delta_{im}\delta_{jn}+\frac{\partial N_{imn}}{\partial y_j}+\frac{\partial N_{jmn}}{\partial y_i}\right]\theta^2_{1y_\beta}\,dy=0\,,\\
&\iint\limits_Y \ell_{ij2\beta}(y)\left[2\delta_{im}\delta_{jn}+\frac{\partial N_{imn}}{\partial y_j}+\frac{\partial N_{jmn}}{\partial y_i}\right]\theta^2_{2y_\beta}\,dy=0\,,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\iint\limits_Y \left[\ell_{ij1\beta}(y)\left(\frac{\partial M_i}{\partial y_j}+\frac{\partial M_j}{\partial y_i}\right)+2\delta_{1\beta}I(y)\right]\theta^2_{1y_\beta}\,dy=0\,,\\
&\iint\limits_Y \left[\ell_{ij2\beta}(y)\left(\frac{\partial M_i}{\partial y_j}+\frac{\partial M_j}{\partial y_i}\right)+2\delta_{2\beta}I(y)\right]\theta^2_{2y_\beta}\,dy=0
\end{aligned} \tag{28}$$

for all $\phi^2 \in H^2_\#(Y)$, $\theta^2_\alpha \in H^1_\#(Y)$, $\alpha = 1, 2$, $(m,n) = (1,1)$, $(2,2)$, $(1,2)$. Note that the cases $(2,1)$ and $(1,2)$ are equivalent.

Proposition 3: *The equation* (25) *is equivalent to the following one:*

$$\gamma(y)\,(1+\Delta_y N) = \langle 1/\gamma\rangle^{-1}\,, \quad \textit{for a.e. } y\in Y\,. \tag{29}$$

The equation (29) *has a unique solution* $N \in H^2_\#(Y)/R$. *The equation* (26) *is equivalent to the following one:*

$$\gamma(y)\,\Delta_y M - I(y) = -\langle 1/\gamma\rangle^{-1}\,\langle I/\gamma\rangle\,, \quad \textit{for a.e. } y\in Y\,. \tag{30}$$

The equation (30) *has a unique solution* $M \in H^2_\#(Y)/R$.

Proof: Let us sketch the proof of (29). The claim (30) can be handled similarly. Note that the equation (25) is equivalent to the following one:

$$\iint\limits_Y [\gamma(y)\,(1+\Delta_y N) - C]\,\Delta_y\phi\,dy = 0\,, \qquad \forall\phi\in H^2_\#(Y)\,, \tag{31}$$

where C is an arbitrary constant. This holds because $\iint\limits_Y \Delta_y\phi\,dy = 0$ for all $\phi \in H^2_\#(Y)$. Let $C = \iint\limits_Y \gamma(y)\,(1+\Delta_y N)\,dy$. Then the equation

$$\Delta_y\phi = \gamma(y)\,(1+\Delta_y N) - C$$

is solvable in $H^2_\#(Y)/R$ because $\iint\limits_Y [\gamma(y)\,(1+\Delta_y N) - C]\,dy = 0$. Substituting the solution into (31) yields

$$\gamma(y)\,(1+\Delta_y N) = C \quad \text{for a.e. } y\in Y\,.$$

Dividing this equation over $\gamma(y)$, integrating over Y, and taking into account that $\iint\limits_Y \Delta_y N\,dy = 0$, we obtain that $C = \langle 1/\gamma\rangle^{-1}$, which proves (29). See [12] for more details if necessary. □

Proposition 4: *Systems* (27), (28) *are uniquely solvable:* N_{imn}, $M_i \in H^1_\#(Y)/R$.

Proof: Let us consider the bilinear form defined on $[H^1_\#(S)/R]^2 \times [H^1_\#(S)/R]^2$ that corresponds to both (27) and (28):

$$\pi(N,L) = \iint\limits_Y \ell_{ij\alpha\beta}\left(\frac{\partial N_i}{\partial y_j}+\frac{\partial N_j}{\partial y_i}\right)\frac{\partial L_\alpha}{\partial y_\beta}\,dy = \frac{1}{2}\iint\limits_Y \ell_{ij\alpha\beta}\left(\frac{\partial N_i}{\partial y_j}+\frac{\partial N_j}{\partial y_i}\right)\left(\frac{\partial L_\alpha}{\partial y_\beta}+\frac{\partial L_\beta}{\partial y_\alpha}\right)dy\,.$$

The form is symmetric and continuous on $[H^1_\#(S)/R]^2 \times [H^1_\#(S)/R]^2$. It follows from (7) that

$$\pi(N,N) \ge \frac{\nu}{2}\iint\limits_Y r_{ij}r_{ij}\,dy\,, \quad \text{where} \quad r_{ij} = \left(\frac{\partial N_i}{\partial y_j}+\frac{\partial N_j}{\partial y_i}\right).$$

Modifying arguments of the proof of Korn's inequality (see [13]), we obtain

$$\pi(N,N) \ge C\iint\limits_Y N_{ix_j}N_{ix_j}\,dy = C\cdot\|N\|^2_{[H^1_\#(S)/R]^2}\,.$$

The linear forms

$$\mu_{mn}(L) = -2\iint\limits_Y \ell_{ij\alpha\beta}\delta_{im}\delta_{jn}\,\frac{\partial L_\alpha}{\partial y_\beta}\,dy\,, \qquad \mu(L) = -2\iint\limits_Y \delta_{\alpha\beta}I(y)\,\frac{\partial L_\alpha}{\partial y_\beta}\,dy$$

corresponding to (27) and (28) are defined and continuous on $[H^1_\#(S)/R]^2$. Applying the Lax-Milgram lemma completes the proof. □

After substituting the functions $\bar{\xi}$, $\bar{u}_\alpha$, $\alpha = 1,2$, (see (23) and (24)) into the effective system (21), we obtain

$$\int_0^T\iint_S \Big\{ \langle\rho\rangle\, \xi\eta_{tt} + \langle\mu\rangle\, \nabla\xi\nabla\eta_{tt} + \langle\gamma[1+\Delta_y N]\rangle\, \Delta\xi\, \Delta\eta + \Big\langle \ell_{ij\alpha\beta}(y) \Big(\delta_{im}\delta_{jn} + \frac{\partial N_{imn}}{\partial y_j} + \frac{\partial N_{jmn}}{\partial y_i}\Big)\Big\rangle\, d_{mn}(\xi,u)\, \xi_{x_\beta}\eta_{x_\alpha} + FK(t,x)\, \langle\gamma\Delta_y M - I\rangle\, \Delta\eta\, dx\, dt + GK(t,x)\, \Big\langle \ell_{ij\alpha\beta}(y) \Big(\frac{\partial M_i}{\partial y_j} + \frac{\partial M_j}{\partial y_i}\Big) + \delta_{\alpha\beta} I(y) \Big\rangle\, \xi_{x_\beta}\eta_{x_\alpha} \Big\} dx\, dt + \iint_S \{\langle\rho\rangle\, (\xi_0\eta_t(0,x) - \xi_0'\eta(0,x)) + \langle\mu\rangle\, (\nabla\xi_0\nabla\eta_t(0,x) - \nabla\xi_0'\nabla\eta(0,x))\}\, dx = 0\,, \tag{32}$$

$$\int_0^T\iint_S \Big\{ \langle\rho\rangle\, u_\alpha\chi_{\alpha tt} + \Big\langle \ell_{ij\alpha\beta}(y) \Big(\delta_{im}\delta_{jn} + \frac{\partial N_{imn}}{\partial y_j} + \frac{\partial N_{jmn}}{\partial y_i}\Big)\Big\rangle\, d_{mn}(\xi,u)\, \chi_{\alpha x_\beta} + GK(t,x)\, \Big\langle \ell_{ij\alpha\beta}(y) \Big(\frac{\partial M_i}{\partial y_j} + \frac{\partial M_j}{\partial y_i}\Big) + \delta_{\alpha\beta} I(y) \Big\rangle\, \chi_{\alpha x_\beta} \Big\}\, dx\, dt + \iint_{S_P} \langle\rho\rangle\, (u_{\alpha 0}\chi_{\alpha t}(0,x) - u_{\alpha 0}'\chi_\alpha(0,x))\, dx = 0\,.$$

Thus, the following system with constant coefficients is obtained:

$$\int_0^T\iint_S \{\hat\rho\xi\eta_{tt} + \hat\mu\nabla\xi\nabla\eta_{tt} + \hat\gamma\Delta\xi\,\Delta\eta + \hat\tau_{\alpha\beta}\xi_{x_\beta}\eta_{x_\alpha} - FK(t,x)\,\hat I\,\Delta\eta + GK(t,x)\,\hat J_{\alpha\beta}\xi_{x_\beta}\eta_{x_\alpha}\}\, dx\, dt + \iint_S \{\hat\rho(\xi_0\eta_t(0,x) - \xi_0'\eta(0,x)) + \hat\mu(\nabla\xi_0\nabla\eta_t(0,x) - \nabla\xi_0'\nabla\eta(0,x))\}\, dx = 0\,, \tag{33}$$

$$\int_0^T\iint_S \{\hat\rho u_\alpha\chi_{\alpha tt} + \hat\tau_{\alpha\beta}\chi_{\alpha x_\beta} + GK(t,x)\,\hat J_{\alpha\beta}\chi_{\alpha x_\beta}\}\, dx\, dt + \iint_{S_P} \hat\rho(u_{\alpha 0}\chi_{\alpha t}(0,x) - u_{\alpha 0}'\chi_\alpha(0,x))\, dx = 0\,.$$

The classical form reads

$$\begin{aligned} &\hat\rho\xi_{tt} - \hat\mu\,\Delta\xi_{tt} + \hat\gamma\,\Delta^2\xi - \frac{\partial}{\partial x_\alpha}(\hat\tau_{\alpha\beta}\xi_{x_\beta}) = F\hat I\,\Delta K(t,x) + G\hat J_{\alpha\beta}\,\frac{\partial}{\partial x_\alpha}\,(\xi_{x_\beta}K(t,x))\,,\\ &\hat\rho u_{\alpha tt} - \frac{\partial}{\partial x_\beta}\,\hat\tau_{\alpha\beta} = G\hat J_{\alpha\beta}\,\frac{\partial}{\partial x_\beta}\,K(t,x)\,, \quad \text{where} \quad \hat\tau_{\alpha\beta} = \hat\ell_{ij\alpha\beta}(u_{ix_j} + u_{jx_i} + \xi_{x_i}\xi_{x_j})\,. \end{aligned} \tag{34}$$

The boundary and initial conditions are

$$\begin{aligned} &\xi|_{\partial S} = 0\,, \qquad \partial\xi/\partial\vec n|_{\partial S} = 0\,, \qquad \xi|_{t=0} = \xi_0\,, \qquad \xi_t|_{t=0} = \xi_0'\,,\\ &u_\alpha|_{\partial S} = 0\,, \qquad u_\alpha|_{t=0} = u_{\alpha 0}\,, \qquad u_{\alpha t}|_{t=0} = u_{\alpha 0}'\,. \end{aligned}$$

Comparing (32) and (33) and using (29) and (30), we obtain explicitly

$$\hat\rho = \langle\rho\rangle\,, \qquad \hat\mu = \langle\mu\rangle\,, \qquad \hat\gamma = \langle 1/\gamma\rangle^{-1}\,, \qquad \hat I = \langle 1/\gamma\rangle^{-1}\langle I/\gamma\rangle\,.$$

Moreover, comparing (32) and (33) gives

$$\hat\ell_{mn\alpha\beta} = \Big\langle \ell_{ij\alpha\beta}(y) \Big[\delta_{im}\delta_{jn} + \frac{1}{2}\Big(\frac{\partial N_{imn}(y)}{\partial y_j} + \frac{\partial N_{jmn}(y)}{\partial y_i}\Big)\Big]\Big\rangle\,, \tag{35}$$

$$\hat J_{\alpha\beta} = \Big\langle \ell_{ij\alpha\beta}(y)\,\frac{1}{2}\Big(\frac{\partial M_i(y)}{\partial y_j} + \frac{\partial M_j(y)}{\partial y_i}\Big) + \delta_{\alpha\beta} I(y)\Big\rangle\,. \tag{36}$$

Conjecture: *The tensor $\hat\ell_{mn\alpha\beta}$ is positive definite, i.e., there is a positive constant ν such that*

$$\hat\ell_{mn\alpha\beta} d_{ij} d_{\alpha\beta} \geq \nu d_{ij} d_{ij} \tag{37}$$

for any symmetric $d \in R^{2\times 2}$.

Unfortunately, we were not able to succeed in formal proving this property. On the other hand, numerous computations show its validity for a wide range of values of σ_B, σ_P, and E_B/E_P (the auxiliary functions depend on the ratio E_B/E_P) that cover all realistic materials. Further, the property (37) is assumed to be valid.

The next theorem follows immediately from the homogenization procedure.

Theorem 1: *Let* $(\xi^{\varepsilon_k}, u_1^{\varepsilon_k}, u_2^{\varepsilon_k})$ *be a sequence of weak solutions of* (5) *converging to some* (ξ, u_1, u_2) *in the weak$*$ topology of* X_∞^{211} *(note that such a sequence always exists due to Proposition 2). Then* (ξ, u_1, u_2) *is a weak solution to problem* (34).

Definition 2: We say that functions (ξ, u_1, u_2) form a strong solution to system (34) if the following holds:

$$\begin{aligned}
&(\xi, u_1, u_2) \in C([0,T]; H_0^2(S)) \times C([0,T]; H_0^1(S)) \times C([0,T]; H_0^1(S)) \,, \\
&(\xi_t, u_{1t}, u_{2t}) \in C([0,T]; H_0^1(S)) \times C([0,T]; L_2(S)) \times C([0,T]; L_2(S)) \,, \\
&(\xi_{tt}, u_{1tt}, u_{2tt}) \in L_\infty(0,T; H_0^1(S)) \times L_\infty(0,T; L_2(S)) \times L_\infty(0,T; L_2(S)) \,, \\
&\xi|_{t=0} = \xi_0 \in H_0^2(S) \,, \qquad \xi_t|_{t=0} = \xi_0' \in H_0^1(S), \\
&u_\alpha|_{t=0} = u_{\alpha 0} \in H_0^1(S) \,, \qquad u_{\alpha t}|_{t=0} = u_{\alpha 0}' \in L_2(S) \,,
\end{aligned} \tag{38}$$

and

$$\hat\rho(\xi_{tt}, \varphi) + \hat\mu(\nabla\xi_{tt}, \nabla\varphi) + \hat\gamma(\Delta\xi, \Delta\varphi) + (\hat\tau_{\alpha\beta}\xi_{x_\beta}, \varphi_{x_\alpha}) - F\hat I(K, \Delta\varphi) + G\hat J_{\alpha\beta}(K\xi_{x_\beta}, \varphi_{x_\alpha}) = 0 \,, \tag{39}$$

$$\hat\rho(u_{\alpha tt}, \psi_\alpha) + (\hat\tau_{\alpha\beta}, \psi_{\alpha x_\beta}) + G\hat J_{\alpha\beta}(K, \psi_{\alpha x_\beta}) = 0 \tag{40}$$

for any $\varphi \in H_0^2(S)$, $\psi_\alpha \in H_0^1(S)$, and a.e. $t \in [0,T]$.

Lemma 1: *If* $K \in L_\infty(0,T; H^1(S))$, *then any strong solution* (ξ, u_1, u_2) *of* (34) *possesses the property*

$$\xi \in L_\infty(0,T; H^3(S)) \,, \qquad u_1, u_2 \in L_\infty(0,T; H^2(S)))\,. \tag{41}$$

Proof: The proof is just similar to the one of Lemma 5.2 of [7]. Namely, we can rewrite (40) as follows:

$$\hat\ell_{ij\alpha\beta}(e_{ij}(u), e_{\alpha\beta}(\psi)) = -\hat\rho(u_{\alpha tt}, \psi_\alpha) - \hat\ell_{ij\alpha\beta}(\xi_{x_i}\xi_{x_j}, e_{\alpha\beta}(\psi)) - G\hat J_{\alpha\beta}(K_{x_\beta}, \psi_\alpha)\,. \tag{42}$$

Using integration by parts and the properties (38), one can prove that $(\xi_{x_i}, \xi_{x_j}, e_{\alpha\beta}(\psi))$ is from $C([0,T]; (L_q(S) \times L_q(S))')$ for any $q > 2$. Therefore, $(\xi_{x_i}, \xi_{x_j}, e_{\alpha\beta}(\psi))$ is from $C([0,T]; H^{-s}(S) \times H^{-s}(S))$ for any $0 < s < 1$ because $H^s(S) \subset L_{2/(1-s)}(S)$ in two dimensions. The other terms on the right-hand side are from $L_\infty(0,T;\, (L_2(S) \times L_2(S))')$. Therefore (see e.g. [11]), u_1 and u_2 are from $L_\infty(0,T; H^{2-s}(S))$, so there exists a constant $C > 0$ such that

$$\|e_{ij}(u)\|_{H^{1-s}} \le C \,, \qquad t \in [0,T]\,.$$

Hence,

$$\begin{aligned}
\|d_{ij}(\xi, u)\, \xi_{x_\beta}\| &= \|e_{ij}(u)\, \xi_{x_\beta} + \tfrac12 \xi_{x_i}\xi_{x_j}\xi_{x_\beta}\| \le \|e_{ij}(u)\|_{L_4(S)}\, \|\xi_{x_\beta}\|_{L_4(S)} + \|\xi_{x_\beta}\|_{L_6(S)}^3 \\
&\le C_1(\|e_{ij}(u)\|_{H^{1-s}(S)}\, \|\xi\|_{H_0^2(S)} + \|\xi\|_{H_0^2(S)}^3) \le C_2 \,,
\end{aligned} \tag{43}$$

for $t \in [0,T]$, if we take $0 < s < 1/2$. Now, rewrite (39) as follows:

$$\hat\gamma(\Delta\xi, \Delta\varphi) \;=\; -\hat\rho(\xi_{tt}, \varphi) - \hat\mu(\nabla\xi_{tt}, \nabla\varphi) - \hat\ell_{ij\alpha\beta}(d_{ij}(\xi, u)\, \xi_{x_\beta}, \varphi_{x_\alpha}) + F\hat I(\nabla K, \nabla\varphi) - G\hat J_{\alpha\beta}(K\xi_{x_\beta}, \varphi_{x_\alpha})\,.$$

Note (38) and (43) imply that all terms on the right-hand-side are from $L_\infty(0,T; H^{-1}(S))$. Therefore (see e.g. [11]), ξ is from $L_\infty(0,T; H^3(S))$. The last result implies that the right-hand-side of (42) is from $L_\infty(0,T; (L_2(S) \times L_2(S))')$. Therefore u_1 and u_2 are from $L_\infty(0,T; H^2(S))$. Thus (41) is proved. □

Lemma 2: *If* $K \in L_\infty(0,T; H^1(S))$, *then the strong solution of* (34) *is unique whenever it exists.*

Proof: Assume that there are two strong solutions ξ^1, u_α^1 and ξ^2, u_α^2. Then the difference $\bar\xi = \xi^1 - \xi^2$, $\bar u_\alpha = u_\alpha^1 - u_\alpha^2$ satisfies the equations

$$\hat\rho(\bar\xi_{tt}, \varphi) + \hat\mu(\nabla\bar\xi_{tt}, \nabla\varphi) + \hat\gamma(\Delta\bar\xi, \Delta\varphi) + \hat\ell_{ij\alpha\beta}(d_{ij}^1\xi_{x_\beta}^1 - d_{ij}^2\xi_{x_\beta}^2, \varphi_{x_\alpha}) + \hat J_{\alpha\beta}(K\bar\xi_{x_\beta}, \varphi_{x_\alpha}) = 0 \,, \tag{44}$$

$$\hat\rho(\bar u_{\alpha tt}, \psi_\alpha) + \hat\ell_{ij\alpha\beta}(d_{ij}^1 - d_{ij}^2, e_{\alpha\beta}(\psi)) = 0 \tag{45}$$

for any $\varphi \in H_0^2(S)$, $\psi_\alpha \in H_0^1(S)$, and for a.e. $t \in [0,T]$. We set $\varphi = (\bar\xi(t+h) - \bar\xi(t-h))/2h$, $\psi_\alpha = (\bar u_\alpha(t+h) - \bar u_\alpha(t-h))/2h$ and integrate from h to $t-h$. The passage to the limit as $h \to 0$ gives

$$\begin{aligned}
&\hat\rho\|\bar\xi_t\|^2 + \hat\mu\|\nabla\bar\xi_t\|^2 + \hat\gamma\|\Delta\bar\xi\|^2 + \hat\rho\|\bar u_{\alpha t}\|^2 + \hat\ell_{\alpha\beta ij}(e_{\alpha\beta}(\bar u), e_{ij}(\bar u)) \\
&\le C \int_0^t (\|d_{ij}^1\xi_{x_\beta}^1 - d_{ij}^2\xi_{x_\beta}^2\|^2 + \|K\bar\xi_{x_\beta}\|^2 + \|\nabla\bar\xi_t\|^2 + \|\nabla(\xi_{x_\alpha}^1\xi_{x_\beta}^1 - \xi_{x_\alpha}^2\xi_{x_\beta}^2)\|^2 + \|\bar u_{\alpha t}\|^2)\, dt\,.
\end{aligned} \tag{46}$$

Using (38), (41) and embedding theorems, we obtain that

$$\begin{aligned}
&\|d^1_{ij}\xi^1_{x_\beta} - d^2_{ij}\xi^2_{x_\beta}\|^2 + \|\nabla(\xi^1_{x_\alpha}\xi^1_{x_\beta} - \xi^2_{x_\alpha}\xi^2_{x_\beta})\|^2 \\
&\le C(\|u^1_{ix_j}\xi^1_{x_\beta} - u^2_{ix_j}\xi^2_{x_\beta}\|^2 + \|\xi^1_{x_i}\xi^1_{x_j}\xi^1_{x_\beta} - \xi^2_{x_i}\xi^2_{x_j}\xi^2_{x_\beta}\|^2 + \|\xi^1_{x_\alpha}\xi^1_{x_\beta x_\gamma} - \xi^2_{x_\alpha}\xi^2_{x_\beta x_\gamma}\|^2) \\
&\le C(\|u^1_{ix_j}(\xi^1_{x_\beta} - \xi^2_{x_\beta})\|^2 + \|\xi^2_{x_\beta}(u^1_{ix_j} - u^2_{ix_j})\|^2 + \|\xi^1_{x_i}\xi^1_{x_j}(\xi^1_{x_\beta} - \xi^2_{x_\beta})\|^2 \\
&\qquad + \|\xi^1_{x_\alpha}(\xi^1_{x_\beta x_\gamma} - \xi^2_{x_\beta x_\gamma})\|^2 + \|\xi^1_{x_\beta x_\gamma}(\xi^1_{x_\alpha} - \xi^2_{x_\alpha})\|^2) \\
&\le C(\|u^1_{ix_j}\|^2_{L_4(S)}\,\|\xi^1_{x_\beta} - \xi^2_{x_\beta}\|^2_{L_4(S)} + \|\xi^2_{x_\beta}\|^2_{L_\infty}\,\|u^1_{ix_j} - u^2_{ix_j}\|^2 + \|\xi^1_{x_i}\xi^1_{x_j}\|^2_{L_4(S)}\,\|\xi^1_{x_\beta} - \xi^2_{x_\beta}\|^2_{L_4(S)} \\
&\qquad + \|\xi^1_{x_\alpha}\|^2_{L_\infty}\,\|\xi^1_{x_\beta x_\gamma} - \xi^2_{x_\beta x_\gamma}\|^2 + \|\xi^1_{x_\beta x_\gamma}\|^2_{L_4(S)}\,\|\xi^1_{x_\alpha} - \xi^2_{x_\alpha}\|^2_{L_4(S)}) \\
&\le C(\|\bar\xi\|^2_{H^2_0(S)} + \|\bar u_\alpha\|^2_{H^1_0(S)}), \qquad t \in [0,T]\,. \qquad (47)
\end{aligned}$$

It is easy to see that

$$\|K\bar\xi_{x_\beta}\|^2 \le \|K\|^2_{L_4(S)}\,\|\bar\xi_{x_\beta}\|^2_{L_4(S)} \le C\|K\|^2_{H^1(S)}\,\|\bar\xi\|^2_{H^2_0(S)} \le C\|\bar\xi\|^2_{H^2_0(S)}\,, \qquad t \in [0,T]\,. \qquad (48)$$

From (46), (47), (48), (2), and the inequality of Korn, we get

$$\|\bar\xi\|^2_{H^2_0(S)} + \|\bar\xi_t\|^2_{H^1_0(S)} + \|\bar u_\alpha\|^2_{H^1_0(S)} + \|\bar u_{\alpha t}\|^2 \le \int_0^t C(\|\bar\xi\|^2_{H^2_0(S)} + \|\bar\xi_t\|^2_{H^1_0(S)} + \|\bar u_\alpha\|^2_{H^1_0(S)} + \|\bar u_{\alpha t}\|^2)\,dt\,.$$

Taking into account that $\bar\xi(0) = 0,\ \bar u_\alpha(0) = 0$, we obtain $\bar\xi = 0,\ \bar u_\alpha = 0$, which proves Lemma 2.

The next theorem states a more useful relation between (5) and (34) than Theorem 1.

Theorem 2: *Let $\xi_0 = 0,\ \xi'_0 = 0,\ u_{\alpha 0} = 0,\ u'_{\alpha 0} = 0,\ K \in H^2(0,T;L_2(S)) \cap L_\infty(0,T;H^1(S))$, and $K|_{t=0} = 0$, then (34) has a unique strong global solution (ξ, u_1, u_2). The sets L^ε shrink to (ξ, u_1, u_2)in weak$*$ topology of X^{211}_∞. That is, any sequence $(\xi^\varepsilon, u^\varepsilon_1, u^\varepsilon_2) \in L^\varepsilon$ converges to (ξ, u_1, u_2) in weak$*$ topology of X^{211}_∞ as $\varepsilon \to 0$.*

Proof: Using Proposition 2, we conclude that there exists a sequence $(\xi^{\varepsilon_k},\ u^{\varepsilon_k}_1,\ u^{\varepsilon_k}_2) \in L^{\varepsilon_k}$ converging to some triple (ξ, u_1, u_2) in weak$*$ topology of X^{211}_∞ as $\varepsilon_k \to 0$. Theorem 1 claims that (ξ, u_1, u_2) is a weak solution of (34). Moreover,

$$(\xi,\ u_1,\ u_2) \in X^{211}_\infty\,, \qquad (\xi_t,\ u_{1t},\ u_{2t}) \in X^{211}_\infty\,, \qquad (\xi_{tt},\ u_{1tt},\ u_{2tt}) \in X^{100}_\infty \qquad (49)$$

due to Proposition 2. Using Theorem 16.1 of [11], we obtain that (ξ, u_1, u_2) possesses the properties (38). Therefore, (ξ, u_1, u_2) is a strong solution of (34). It is unique due to Lemma 2.

Assume that L^ε does not shrink to (ξ, u_1, u_2). Then one can find a subsequence $(\xi^{\varepsilon_k}, u^{\varepsilon_k}_1, u^{\varepsilon_k}_2) \in L^{\varepsilon_k}$ separated from (ξ, u_1, u_2) in the weak$*$ topology of X^{211}_∞. There exists a sub-subsequence $(\xi^{\varepsilon_{k_l}}, u^{\varepsilon_{k_l}}_1, u^{\varepsilon_{k_l}}_2) \in L^{\varepsilon_{k_l}}$ that converges to some (ξ^*, u^*_1, u^*_2) in the weak$*$ topology of X^{211}_∞. The same arguments as above yield that (ξ^*, u^*_1, u^*_2) is a strong solution of (34). Hence, $(\xi^*, u^*_1, u^*_2) = (\xi, u_1, u_2)$, which is a contradiction. □

Remark 2: It is sufficient to assume that $\xi_0 \in H^2_0(S) \cap H^3(S),\ \xi'_0 \in H^2_0(S),\ u_{\alpha 0} \in H^1_0(S) \cap H^2(S),\ u'_{\alpha 0} \in H^1_0(S)$, $K \in H^2(0,T;L_2(S)) \cap L_\infty(0,T;H^1(S))$ for the existence of strong global solutions of (34). The prove is based on the time differentiation of a relationship defining Galerkin approximations ξ^m, u^m_1, u^m_2 for (34). This yields an equation that defines the time derivatives $w^m = \xi^m_t,\ v^m_1 = u^m_{1t},\ v^m_2 = u^m_{2t}$, if the initial values for $w^m,\ v^m_1,\ v^m_2$ are chosen properly. Due to constant coefficients of (34), the above assumptions ensure compatibility conditions (see e.g. [9]) that make possible to find such appropriate initial values for $w^m,\ v^m_1,\ v^m_2$. Thus, an estimate like (12) holds for $\xi^m,\ u^m_1,\ u^m_2$. This is sufficient to prove that any limit point of $\{\xi^m,\ u^m_1,\ u^m_2\}$ is a strong global solution of (34).

5. Simulation

The simulation was done with the following parameters: S is the unit square, S_{P_l} is the $\varepsilon/2$ edge square centered w.r.t. the corresponding structural cell (see Fig. 1). The other values are:

$$\begin{aligned}
&\rho_P = 2\,, \quad \rho_B = 1\,, \quad \gamma_P = 5\,, \quad \gamma_B = 3\,, \quad \mu_P = 0.1\,, \quad \mu_B = 0.1\,, \\
&E_P = 2\,, \quad E_B = 1\,, \quad \sigma_P = 0.4\,, \quad \sigma_B = 0.2\,, \quad \varepsilon = 1/6\,, \\
&F = 5\,, \quad G = 5\,, \quad K(t, x_1, x_2) = \sin 5t \cos 5(x_1 + x_2)\,.
\end{aligned}$$

The initial values were equal to zero. The Bogner-Fox-Schmit finite elements (see CIARLET [14]) were applied. The number of finite elements was equal to 24×24. Each structural cell (see Fig. 1) occupies 4×4 elements. Each piezopatch occupies 2×2 elements. Each figure below shows the solution ξ^ε of (5), the solution ξ of (34), and their difference at times indicated.

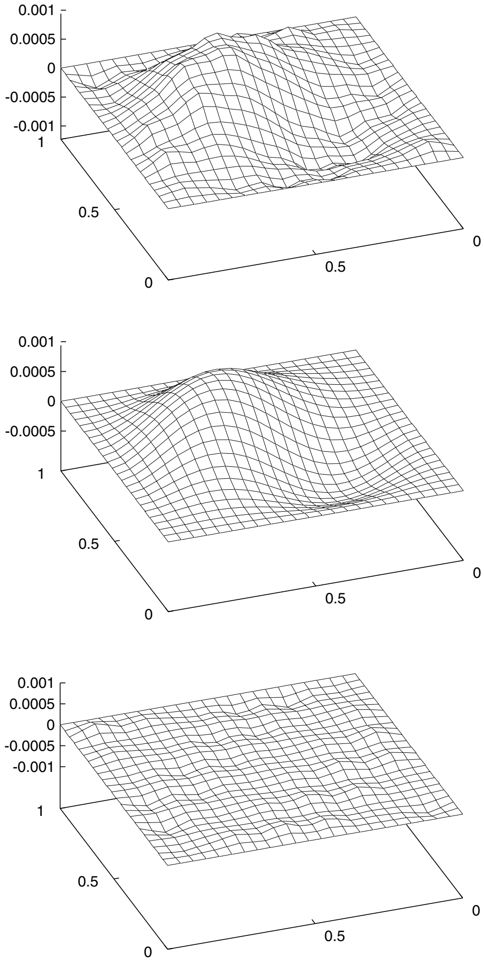


Fig. 2. $t = 0.7$

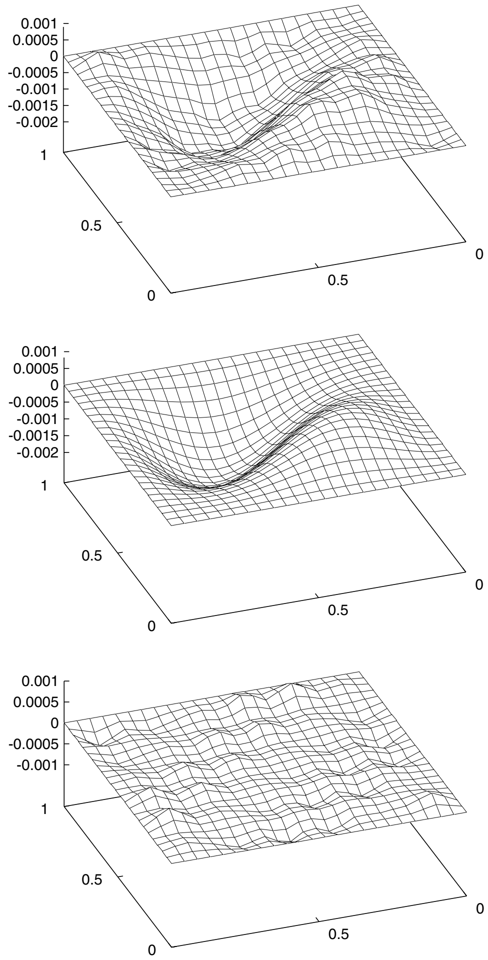


Fig. 3. $t = 1.4$

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