# On the Marcinkiewicz and $(C, \alpha)$-means of the quadratical partial sums of double Walsh-Kaczmarz-Fourier series

Károly Nagy

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Inzell

K. Nagy,
Institute of Mathematics and Computer Science,
College of Nyíregyháza, P.O. Box 166,
Nyíregyháza, H-4400
Hungary

*E-mail address:* nkaroly@nyf.hu

# Introduction

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, the group operation is the modulo 2 addition, every subset is open. Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2.

The Walsh group:

$$G := \mathop{\times}_{k=0}^{\infty} Z_2.$$

The elements of $G$ are of the form

$$x = (x_0, x_1, ..., x_k, ...)$$

with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$.

The group operation on $G$ is the coordinate-wise addition, the measure (denoted by $\mu$) and the topology are the product measure and topology.

Rademacher functions:

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Every $n \in \mathbf{N}$ can be written in the form

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \ (i \in \mathbf{N}).$$

Let be the order of $n$

$$|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}.$$

Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|-1} n_k x_k}$$

Walsh-Paley system: $\omega := (\omega_n : n \in \mathbf{N})$

Walsh-Kaczmarz functions:

$$\begin{aligned} \kappa_n(x) \quad &:= \quad r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} \\ &= \quad r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}, \end{aligned}$$

The Walsh-Kaczmarz system: $\kappa := (\kappa_n : n \in \mathbf{N})$.

It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{N}$ and $\kappa_0 = \omega_0$.

A relation between Walsh-Kaczmarz functions and Walsh-Paley functions :

The transformation $\tau_A : G \to G$ $(A \in \mathbf{N})$ defined by V. A. Skvortsov

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

gives

$$\kappa_n(x) = r_{|n|}(x)\omega_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels:

$$\hat{f}^{\psi}(n) := \int_G f\bar{\psi}_n, \;\; S_n^{\psi} f := \sum_{k=0}^{n-1} \hat{f}^{\psi}(k)\psi_k,$$
$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k, \quad \sigma_n^{\psi} f := \frac{1}{n}\sum_{k=0}^{n-1} S_k^{\psi} f,$$
$$K_n^{\psi} := \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\psi}.$$

where $\psi_n = \omega_n$ or $\kappa_n$.

The two-dimensional Walsh group: $G \times G$,

the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels, the Marcinkiewicz means and Marcinkiewicz kernels:

$$\hat{f}^{\psi}(n_1, n_2) := \int_{G\times G} f\bar{\psi}_{n_1}\bar{\psi}_{n_2},$$
$$S_{n_1,n_2}^{\psi} f(x^1, x^2) := \sum_{k=0}^{n_1-1}\sum_{l=0}^{n_2-1} \hat{f}^{\psi}(k,l)\psi_k(x^1)\psi_l(x^2),$$
$$D_{n_1,n_2}^{\psi}(x^1, x^2) := D_{n_1}^{\psi}(x^1)D_{n_2}^{\psi}(x^2),$$
$$\mathcal{M}_n^{\psi} f := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}^{\psi} f, \quad \mathcal{K}_n^{\psi} := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}^{\psi}.$$

It is well-known that:

$$S_{\underline{n}}^{\psi} f(\underline{y}) = \int_{G\times G} f(\underline{x}+\underline{y})D_{\underline{n}}^{\psi}(\underline{x})d\mu(\underline{x}),$$
$$\mathcal{M}_n^{\psi} f(\underline{y}) = \int_{G\times G} f(\underline{x}+\underline{y})K_n^{\psi}(\underline{x})d\mu(\underline{x}),$$
$$(\underline{n} = (n_1, n_2), \underline{y} \in G\times G, \; f \in L^1(G\times G)),$$

where $\psi_n = \omega_n$ or $\kappa_n$.

Define the maximal operator of the two-dimensional Marcinkiewicz means with respect to Walsh-Kaczmarz system

$$\mathcal{M}^{\kappa *} f := \sup_{n\in\mathbf{P}} |\mathcal{M}_n^{\kappa} f|.$$

## Some historical notes on Walsh-Kaczmarz system

- A. A. Šneider(1948):

$$\limsup_{n\to\infty} \frac{D_n^\kappa(x)}{\log n} \geq C > 0 \quad a.e.$$

- F. Schipp and Wo-Sang Young (1974): the Walsh-Kaczmarz system is a convergence system.

- V. A. Skvortsov (1981): the Fejér means converges uniformly to $f$ for any continuous functions $f$.

- G. Gát (1998): the Fejér means converges almost everywhere to the function for any integrable functions.

## Some historical notes on Marcinkiewicz means

- J. Marcinkiewicz (1939): for two-dimensional trigonometric system

$$\mathcal{M}_n f \to f \quad a.e.$$

for all $f \in L\log L([0,2\pi]^2)$.

- L. V. Zhizhiashvili (1968): the same result for all $f \in L([0,2\pi]^2)$.

- U. Goginava (2000), F. Weisz (2001) (independently): $\mathcal{M}_n f \to f$ a. e. for all $f \in L^1([0,1]^2)$ with respect to Walsh-Paley system.

- U. Goginava (2003): $\mathcal{M}_n f \to f$ a. e. for all $f \in L^1$ with respect to the $d$-dimensional Walsh-Paley system.

- G. Gát (2004): $\mathcal{M}_n f \to f$ a. e. for all $f \in L^1(G_m \times G_m)$ where $\mathcal{M}_n f$ defined with respect to two-dimensional Vilenkin systems.

## Results on Walsh-Kaczmarz-Marcinkiewicz means

In the following paper gives a good decomposition of Marcinkiewicz kernels with respect to Walsh-Kaczmarz system, which gives Theorem 1.

K. Nagy, *Some convergence properties of the Walsh-Kaczmarz system with respect to the Marcinkiewicz means,* Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. **76** (2005), 503-516.

**Theorem 1.** *The operator $\mathcal{M}^{\kappa *}$ is of type $(\infty,\infty)$, that is there exists a constant $c>0$, such that*

$$\|\mathcal{M}^{\kappa *} f\|_\infty \leq c\|f\|_\infty$$

*holds for all $f \in L^\infty(G\times G)$.*

In the paper

K. Nagy, *On the two-dimensional Marcinkiewicz means with respect to Walsh-Kaczmarz system,* Journal of Approximation Theory **142** (2006) 138-165.

**Theorem 2.** *The maximal function $\mathcal{M}^{\kappa *}$ is of weak type $(1,1)$, that is there exists a constant $c>0$ such that*

$$\mu(\{\underline{y}:\mathcal{M}^{\kappa *}f(\underline{y})>\lambda\})\leq c\frac{\|f\|_1}{\lambda}$$

*holds for all $\lambda>0$ and $f\in L^1(G\times G)$.*

**Theorem 3.** *For all $f\in L^1(G\times G)$*

$$\mathcal{M}^{\kappa}_n f\to f \quad a.e.$$

*relation holds.*

By the Marcinkiewicz interpolation theorem

**Corollary 1.** *The operator $\mathcal{M}^{\kappa *}$ is of type $(p,p)$ for all $1<p\leq\infty$, that is there exists a constant $c_p>0$ (depends only on $p$), such that*

$$\|\mathcal{M}^{\kappa *}f\|_p\leq c_p\|f\|_p$$

*holds for all $f\in L^p(G\times G)$.*

# Generalization

Set $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$ for any $n\in\mathbf{N},\alpha\in\mathbf{R}$. It is known that $A_n^\alpha\sim n^\alpha$. Let the $(C,\alpha)$ means and kernels, be defined as:

$$t_n^{\psi,\alpha}f:=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}S_{k,k}^{\psi}f,\quad T_n^{\psi,\alpha}:=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}D_{k,k}^{\psi},$$

where $\psi_n=$ either $\omega_n$ or $\kappa_n$ $(n\in\mathbf{P})$.
Thus, the Marcinkiewicz means and Marcinkiewicz kernels is

$$\mathcal{M}_n^{\psi}:=t_n^{\psi,1},\quad \mathcal{K}_n^{\psi}:=T_n^{\psi,1},$$

Define the maximal operator of the $(C,\alpha)$ means $(\alpha>0)$ of a function $f\in L^1(G\times G)$ by

$$t_*^{\psi,\alpha}f:=\sup_{n\in\mathbf{P}}|t_n^{\psi,\alpha}f|.$$

## Some historical notes on $(C, \alpha)$ means of quadratical partial sums

- L. V. Zhizhiashvili (1968): for two-dimensional trigonometric system holds that the $(C, \alpha)$ means converge to $f$ a.e. for any $\alpha > 0$ and for $f \in L([0, 2\pi]^2)$.

- M.I. Daychenko (1988): improved Zhizhiashvili's result for dimensions greater than 2.

- G. Gát, U. Goginava (2006): the $(C, \alpha)$ means converge to $f$ a.e. for any $\alpha > 0$ with respect to two-dimensional bounded Vilenkin system.

## Results on $(C, \alpha)$ means of quadratical partial sums with respect to Walsh-Kacmarz system

In the paper

G. Gát, K. Nagy, *On the $(C, \alpha)$ means of quadratical partial sums of double Walsh-Kaczmarz Fourier series,* Georgian Mathematical Journal (2007) (submited).

**Theorem 4.** *Let $f \in L^p(G \times G)$ and $\alpha > 0$, then the maximal function $t_*^{\kappa,\alpha}$ is of type $(p, p)$ for all $1 < p \le \infty$, that is there exists a constant $c_p > 0$ (depends only on $p$), such that*

$$\|t_*^{\kappa,\alpha} f\|_p \le c_p \|f\|_p$$

*holds for all $f \in L^p(G \times G)$.*

Moreover,

**Theorem 5.** *Let $f \in L^1(G \times G)$ and $\alpha > 0$, then the maximal function $t_*^{\kappa,\alpha}$ is of weak type $(1, 1)$, that is there exists a constant $c > 0$ such that*

$$\mu(\{\underline{y} : t_*^{\kappa,\alpha} f(\underline{y}) > \lambda\}) \le c \frac{\|f\|_1}{\lambda}$$

*holds for all $\lambda > 0$ and $f \in L^1(G \times G)$.*

This gives by standard density argument the following theorem.

**Theorem 6.** *Let $f \in L^1(G \times G)$ and $\alpha > 0$, then*

$$t_n^{\kappa,\alpha} f \to f \text{ a.e. as } n \to \infty.$$

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