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Application of a Dynamic Programming Method to Enlarged Version of Stoleru's Economic Model

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Abstract: This study applies dynamic programming and numerical methods to an enlarged version of Stoleru's economic model of Algeria to achieve optimization with respect to the criteria of full employment and balanced growth afterwards while maintaining a minimum per capita consumption. The model is expanded with new influential quantities including a state constraint, a disturbance factor, and an additional export sector. Results are presented for the case of Algeria in 1960 and 1961.

Keywords: optimal control, dynamic programming, economic, numerical methods, parallel computation, differential equations.

1. INTRODUCTION

This study applies the dynamic programming and the numerical methods presented by Botkin et al. (2011a, b) to an enlarged version of Stoleru's economic model of Algeria (Stoleru 1965) that takes into consideration some new affecting factors. This paper starts with a quick summary of existing Stoleru's closed model. Subsequently, we analyze the case where the country receives external foreign aid and when a state constraint is applied in the form of minimum per capita consumption. Furthermore, we expand the model to consider an additional export sector. When applying the method, we can also consider a disturbance factor that represents the unpredictable events of the surrounding environment that may have some influences on the economy. The numerical method is applied to the numerical data of Algeria that were taken in 1960 and 1961, but the method is more general and can be applied to other economical data. We include in this paper some of the computation results, charts, and comparisons between models.

2. STOLERU'S ECONOMIC MODEL OF ALGERIA

Stoleru (1965) considered a simple closed model with capital and consumption sectors. The goal to be achieved was to determine the optimal way of allocating investment into these two sectors to reach full employment as soon as possible and balanced growth afterward (Goodwin 1961). He assumed that the capital cannot be transferred between sectors and there is no delay between the production of the capital goods and investment.

2.1 Algeria in 1960 and 1961

In 1960 and 1961, Algeria was facing many problems that had major impact on its economical situation. It was characterized by a high growth rate of population (2.2 annual growth rate for a population of 10 million people), a low per capita income (200$ per year for the average citizen), and severe unemployment (40% in 1961) (Stoleru 1965; Yassouridis 2010).

2.2 Mathematical Formulation of Employment Problem

Stoleru (1965) introduced new variables for the two sectors:

Table 1. Sectors variables

	Capital sector	Consumption sector
Output	Y_I	Y_C
Capital	K_I	K_C
Labor force	L_I	L_C
Capital-output ratio	$1/\alpha$	$1/\beta$
Labor-capital ratio	b	c

In both sectors, there is a direct relation between the capital and the output as well as between the capital and the labor force:

$$Y_I = \alpha\, K_I, \qquad L_I = b\, K_I, \tag{1}$$

$$Y_C = \beta\, K_C, \qquad L_C = c\, K_C. \tag{2}$$

Two new variables were introduced by Stoleru (1965):

$$y(t) = \frac{K_I(t)}{K_I(0)}\, e^{\mu t}, \qquad z(t) = \frac{K_C(t)}{K_I(0)}\, e^{\mu t}. \tag{3}$$

Here μ is a constant rate of depreciation occurs in both sectors, $y(t)$ the ratio of capital existing in the capital sector at time t to the stock $K_I(0)e^{-\mu t}$ which would exist if no investment had been made since time 0, and $z(t)$ the same ratio but for the consumption sector. The investment made by the government at any time t in either sector can be written as:

$$\dot{y} = \dot{K_I} + \mu K_I \ , \qquad \dot{z} = \dot{K_C} + \mu K_C \tag{4}$$

(we use the notation $\dot{x} = dx/dt$). The control variable $u(t)$ is defined as:

$$u := \frac{\text{Gross investment in the capital sector}}{\text{Total gross investment}} = \frac{\dot{y}}{\dot{y} + \dot{z}} \ , \tag{5}$$

$u(t)$ represents the portion of the total investment that the government should put into the capital sector. Since we consider only two sectors and the government will not draw money out of the sectors, $u(t)$ will take 1 as its maximum value and 0 as its minimum. To achieve full employment, the labor force should be expanding at the growth rate of the population:

$$L(t) = e^{nt} L_f(0) \, . \tag{6}$$

Here $L(t)$ is the labor force employed in both sectors at time t, $L_f(0)$ is the initial total available labor force, and n is the constant growth rate of population. One can express this condition by the two variables $y(t)$ and $z(t)$:

$$l_0 \, e^{\gamma t} - \left(\delta \, z(t) + y(t)\right) = 0 \, . \tag{7}$$

The new defined constants are:

$$l_0 = \frac{L_f(0)}{L_I(0) = \text{initial labor force employed in the capital sector}} = 11,$$

$\gamma = \mu + n = 0.1 + 0.025 = 0.125$, and $\delta = c/b = 1.8$.

By applying some transformations, the system of full employment is:

$$\dot{y} = \alpha u y \, , \qquad y(0) = 1 \, ,$$
$$\dot{z} = \alpha(1-u) y \, , \qquad z(0) = \sigma = 3.1 \, . \tag{8}$$

The initial values are σ which is the ratio of the initial capital in the consumption sector to the initial capital in the capital sector and $y(0)$ which is the ratio of the initial capital in the capital sector to the initial capital in the capital sector. In the case of Algeria in 1961, α equals 0.25.

The goal is finding the earliest time T at which Algeria can achieve full employment:

$$T = \min_{u(\cdot)} \min\left\{t : l_0 \, e^{\gamma t} - \left(\delta \, z(t) + y(t)\right) = 0\right\}. \tag{9}$$

2.3 Foreign Aid

Algeria was receiving big funds from France which should be taken into consideration to make the model more realistic. Stoleru (1965) assumed that the annual amount is constant. Furthermore, Yassouridis (2010) assumes that foreign aid is available until Algeria reaches full employment and it has a specific limit to keep Algeria independent. Stoleru (1965) added a new variable f to represent the received foreign aid:

$$f := \frac{\text{Annual foreign aid in capital goods}}{\text{Initial domestic production of capital goods}}$$

To satisfy the above assumptions, Yassouridis (2010) assumes that f takes only values between $[1,3]$ which means that the external aid cannot be more than three times as much as the money generated by local production. The system (8) is enlarged by adding f:

$$\dot{y} = \alpha u(y + f e^{\mu t}) \, , \qquad y(0) = 1 \ ,$$
$$\dot{z} = \alpha(1-u)(y + f e^{\mu t}) \, , \qquad z(0) = \sigma = 3.1 \, . \tag{10}$$

2.4 Adding State Constraint

To avoid resentment of population, consumption rate should not be decreased in order to reach the full employment earlier. Stoleru (1965) added the constraint that per capita consumption must not be less than a certain minimum rate m for all time $t \in [0, T]$ where T is full employment time:

$$\text{per capita consumption} = \frac{1}{\sigma} \, z(t) e^{-\gamma t} \geq m \, . \tag{11}$$

3. EXTENDED EXPORT MODEL

3.1 Mathematical formulation of the problem

We add a new *Export* sector and assume that the output of the export sector is additional money that can be distributed by the government between the capital, the consumption, and the export sector. We introduce the following new variables for the export sector: K_E as the existing capital, L_E as the employed labor force, α_x as the output-capital ratio, g as the labor-capital ratio, and $x(t)$ as the ratio of capital existing in the export sector at time t to the stock $K_I(0)e^{-\mu t}$ which would exist if no investment had been made since time 0:

$$x = \frac{K_E(t)}{K_I(0)} \, e^{\mu t} \ . \tag{12}$$

We enlarge the model (10) with the export sector:

$$\dot{x} = u_x\left(\alpha_x x + \alpha_y y + f e^{\mu t}\right) \ , \qquad x(0) = x_0 \, ,$$
$$\dot{y} = u_y\left(\alpha_x x + \alpha_y y + f e^{\mu t}\right) , \qquad y(0) = 1 \, , \tag{13}$$
$$\dot{z} = u_z\left(\alpha_x x + \alpha_y y + f e^{\mu t}\right) , \qquad z(0) = \sigma = 3.1 \, .$$

Here $(\alpha_x x + \alpha_y y)$ is the production of investment that can be achieved, u_i is the fraction of investment into the corresponding sector for $i = x, y, z$ such that $\sum_{i=0}^{3} u_i = 1$, and x_0 is the ratio of the initial capital in the export sector to the initial capital in the capital sector. The condition of full employment for model (13) with capital, consumption, and export sectors can be written as:

$$l_0 \, e^{\gamma t} - \left(\lambda \, \mathrm{x(t)} + \delta \, z(t) + y(t)\right) = 0, \tag{14}$$

where $\lambda = g/b$.

The numerical values of the new variables in the expanded model were not given for the case of Algeria in 1961, so in order to apply our numerical method; we assume some reasonable values for those variables: $\alpha_x = 0.8$, $\lambda = 1.2$, and $x_0 = 0.5$ (we assume that the capital sector is more capital intensive than the export sector). For the remaining variables, we use the data of Algeria in 1961.

4. STATEMENT OF CONTROL PROBLEMS

For convenience, we apply some transformations (we use the "hat" notation, i.e. $\hat{y}$, to represent variables after transformation):

$$\hat{x} = \frac{x}{l_0 e^{\gamma t}} \quad , \quad \hat{y} = \frac{y}{l_0 e^{\gamma t}} \quad , \quad \hat{z} = \frac{z}{l_0 e^{\gamma t}} \quad , \tag{15}$$

$t = 10\,\tau$ and consequently $T = 10\,\hat{T}$.

We apply the transformations on the foreign aid system (10):

$$\hat{y}_\tau^{\prime} = -\,10\gamma\,\hat{y} + 10\alpha u\left(\hat{y} + \frac{f}{l_0}\,e^{(\mu-\gamma)10\tau}\right), \quad \hat{y}_0 = \frac{y_0}{l_0}, \tag{16}$$

$$\hat{z}_\tau^{\prime} = -\,10\gamma\,\hat{z} + 10\alpha(1-u)\left(\hat{y} + \frac{f}{l_0}\,e^{(\mu-\gamma)10\tau}\right), \quad \hat{z}_0 = \frac{z_0}{l_0}.$$

And the minimum time T to achieve full employment with the foreign aid, see (9), becomes:

$$T = 10\,\hat{T}, \text{ where} \tag{17}$$

$$\hat{T} = \min_{u(\cdot)} \min\{t : 1 - (\delta\,\hat{z}(t) + \hat{y}(t)) = 0\} .$$

The state constraint (11) is transformed into:

$$\frac{\sigma\, m}{l_0} - \hat{z}(t) \le 0 . \tag{18}$$

Now we apply the same transformations to extended export model (13):

$$\hat{x}_\tau^{\prime} = -10\gamma\,\hat{x} + 10u_x\left(\alpha_x\hat{x} + \alpha_y\hat{y} + \frac{f}{l_0}\,e^{(\mu-\gamma)10\tau}\right),$$

$$\hat{y}_\tau^{\prime} = -10\gamma\,\hat{y} + 10u_y\left(\alpha_x\hat{x} + \alpha_y\hat{y} + \frac{f}{l_0}\,e^{(\mu-\gamma)10\tau}\right), \tag{19}$$

$$\hat{z}_\tau^{\prime} = -10\gamma\,\hat{z} + 10u_z\left(\alpha_x\hat{x} + \alpha_y\hat{y} + \frac{f}{l_0}\,e^{(\mu-\gamma)10\tau}\right),$$

$$\hat{x}_0 = \frac{x_0}{l_0}, \qquad \hat{y}_0 = \frac{y_0}{l_0}, \qquad \hat{z}_0 = \frac{z_0}{l_0}.$$

And the minimum time to achieve full employment T with extended export model is:

$$T = 10\,\hat{T}, \text{ where} \tag{20}$$

$$\hat{T} = \min_{u_x(\cdot),u_y(\cdot),u_z(\cdot)} \min\{t : 1 - (\lambda\,\hat{x}(t) + \delta\,\hat{z}(t) + \hat{y}(t)) = 0\}.$$

5. THE APPLIED NUMERICAL METHOD

There are many works devoted to computing viscosity solutions of Hamilton-Jacobi equations arising from control problems and differential games, see, e.g. Bard et al. (1999), Bokanowski et al. (2010), Botkin et al. (2011 a, b), Cardaliaguet et al. (1999), Mitchell (2002), Taras'yev et al. (1995). In this work, we apply the numerical method proposed by Botkin et al. (2011a, b) for solving optimal control problems where there is a control $u(t) \in [0,1]$, a criterion to be optimized, and a disturbance $|v(t)| \le \varepsilon$ which is added to the right hand side of the controlled system, for example (the following system is only for illustration purposes and it is not related to the economic models of Algeria described earlier):

$$\begin{aligned} \dot{x} &= 1, & x(0) &= x_0\,, \\ \dot{y} &= \alpha u y + v\,, & y(0) &= 1\,, \\ \dot{z} &= \alpha(1-u)y - v, & z(0) &= \sigma = 3.1\,. \end{aligned} \tag{21}$$

The system starts at $t_0 \in [0,\theta]$ and finishes at θ. The payoff function defined on the trajectories of the system:

$$J(u(\cdot)) = \max \tag{22}$$

$$\left\{ \min_{t\in[t_0,\theta]} G(t,x(t),y(t),z(t)),\ \max_{t\in[t_0,\theta]} \omega(t,x(t),y(t),z(t)) \right\},$$

where $G\colon [0,\theta] \times R^n \to R$ is a given objective function, n is the system dimension, and ω is a function that imposes a state constraint $(\omega(t,x(t),y(t),z(t)) \le 0)$. The value function $V\colon (t,x,y,z) \to V(t,x,y,z)$ is defined by the relation:

$$V(t_0,x_0,y_0,z_0) = \min_{u(t,x,y,z)} \max_{v(\cdot)} J(u(\cdot))\ |_{t_0,x_0,y_0,z_0} . \tag{23}$$

The value function satisfies the following Hamilton–Jacobi–Bellman equation:

$$\dot{V}(t,x,y,z) + \min_{u\in[0,1]} \max_{|v|\le\varepsilon} \nabla V(t,x,y,z)\,.\,\vec{f}\,(t,x,y,z,u) = 0\ , \tag{24}$$

where $a\,.\,b$ means the dot product of the two vectors, ∇ is the gradient operator, and f_i is the ith component of the right hand side of the controlled system. We start the method by forming a 3-dimensional grid. We introduce the following notations:

$\tau, \Delta_x, \Delta_y, \Delta_z$ are the time and space discretization step sizes,

$t_m = m\tau$,

$V^m(x_i,y_j,z_k) = V(t_m, i\,\Delta_x, j\,\Delta_y, k\,\Delta_z)$, $m = 0,\dots,M$ where $M\tau = \theta$.

We move backward in time, starting with the full employment time T and ending with time 0. The backward step is defined as follows:

$$\bar{V}^{m-1}(x_i,y_j,z_k) = V^m(x_i,y_j,z_k) + \tau \min_{u\in\{0\},\{1\}} \max_{|v|\le\varepsilon} \sum_{i=1}^{n} (\,p_i^R\,.\,f_i^+ + p_i^L\,.\,f_i^-\,)\,, \tag{25}$$

where p_i^R, p_i^L are the right and left approximations of the gradient of V^m:

$$p_1^R = [V^m(x_{i+1},y_j,z_k) - V^m(x_i,y_j,z_k)]/\Delta_x\,,$$

$$p_2^R = [V^m(x_i,y_{j+1},z_k) - V^m(x_i,y_j,z_k)]/\Delta_y\,,$$

$$p_3^R = [V^m(x_i,y_j,z_{k+1}) - V^m(x_i,y_j,z_k)]/\Delta_z\,,$$

$$p_1^L = [V^m(x_i,y_j,z_k) - V^m(x_{i-1},y_j,z_k)]/\Delta_x\,,$$

$$p_2^L = [V^m(x_i,y_j,z_k) - V^m(x_i,y_{j-1},z_k)]/\Delta_y\,,$$

$$p_3^L = [V^m(x_i, y_j, z_k) - V^m(x_i, y_j, z_{k-1})]/\Delta_z \ ,$$

and $f_i^{\ +}$, $f_i^{\ -}$ are the positive and negative parts respectively:

$$f_i^{\ +} = \max(f_i, 0), \qquad f_i^{\ -} = \min(f_i, 0).$$

The step is completed by

$$V^{m-1}(x_i, y_j, z_k) = \max\{A, B\}, \tag{26}$$

where

$A := \min\{\bar{V}^{m-1}(x_i, y_j, z_k), G(t_{m-1}, x_i, y_j, z_k)\}$,

$B := \omega(t_{m-1}, x_i, y_j, z_k)$.

Note that

$m = M, M\text{-}1, \cdots, 1$, and $V^M(x_i, y_j, z_k) = G(\theta, x_i, y_j, z_k)$.

At each time step, we calculate the value function at each point of the grid and the minimizing u at that point and store them to compute optimal trajectories. The time of full employment T is:

$$T = \min\{\theta : V_\theta(0, x_0, y_0, z_0) = 0\}. \tag{27}$$

6. RESULTS

The numerical method described above is appropriate for parallelization on multiprocessor computers. All results presented in this paper are calculated using programs written in C programming language and OpenMP (Open Multi-Processing) for parallelization.

6.1 Calculating Full Employment Time of Foreign Aid System

We calculate the full employment time T for the foreign aid system (10) with different values of foreign aid factor f ($f \in [1,3]$) using transformed system (16) as follows:

$$\hat{T} = \min\{\hat{\theta} : \hat{V}_{\hat{\theta}}(0, \hat{y}_0, \hat{z}_0) = 0\}, \tag{28}$$

where

$$\hat{V}_{\hat{\theta}} = \min_{u(\cdot)} \min_{\tau \in [0,\hat{\theta}]} (1 - \delta\, \hat{z}(\tau) - \hat{y}(\tau)) \ \ and\ \ T = 10\,\hat{T}\,.$$

The value function is calculated with a grid of size 700×700, the time step equals 5×10^{-4}, and the number of parallelization threads is 30. In Figure 1, one can see that the more foreign aid received by Algeria the earlier the full employment is reached (no disturbances are considered).

Next, we calculate the full employment time T for the same foreign aid system but with considering a disturbance as in (21), where $|v(t)| \le \varepsilon = 0.1$ (see Fig. 2). Comparing between Figure 1 and Figure 2, one can see that the full employment time T is larger after adding disturbances.

Now we add the state constraint (11) to the foreign aid system (10) and calculate the full employment time T, using transformed state constraint (18) and transformed system (16) as in (28), for different values of minimum per capita consumption m ($m \in [0.1,1]$) but without considering any disturbances (see Fig. 3). One can see that at any given value of the foreign aid f, the larger is the minimum per capita consumption m the more time is required to achieve full employment.

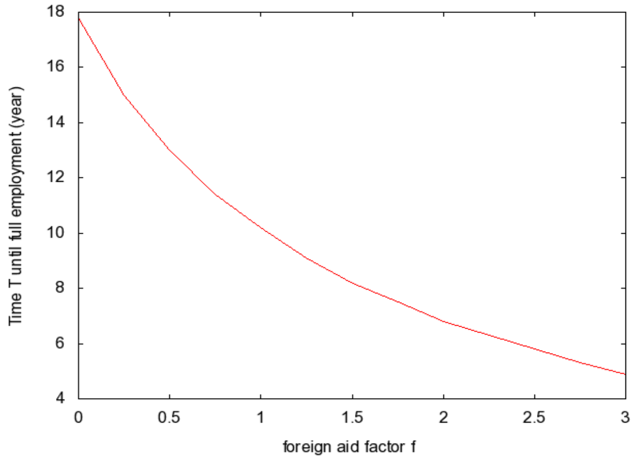

Fig. 1. The relation between foreign aid f and full employment time T without disturbances.

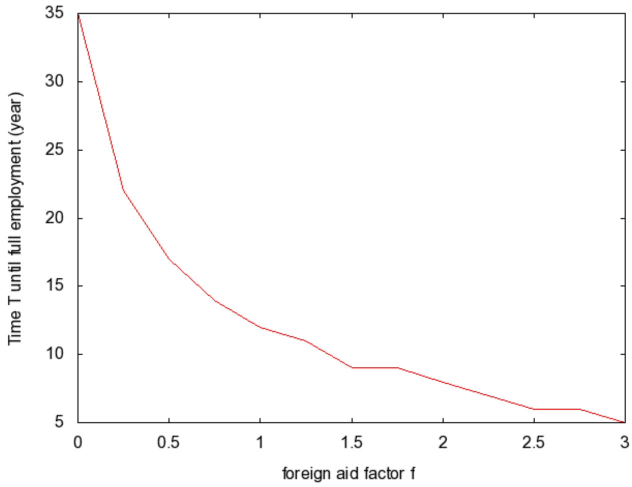

Fig. 2. The relation between foreign aid f and full employment time T with disturbances.

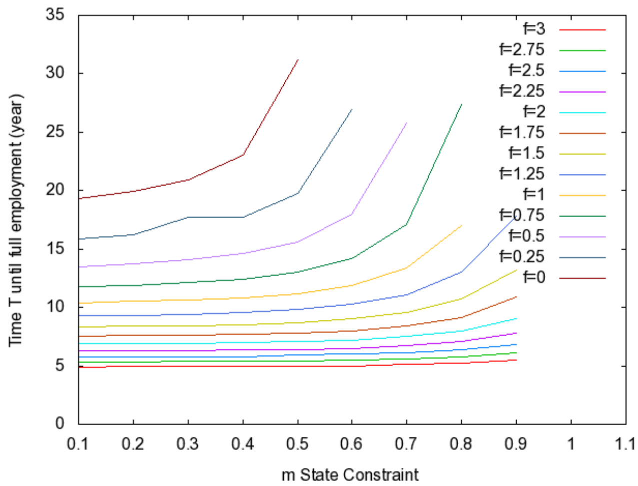

Fig. 3. The relation between foreign aid f, full employment time T, and minimum per capita consumption m without disturbances.

Here, we calculate the full employment time T for the same system as in Figure 3 but additionally we consider a disturbance, and as expected the values of T increased (see Fig. 4).

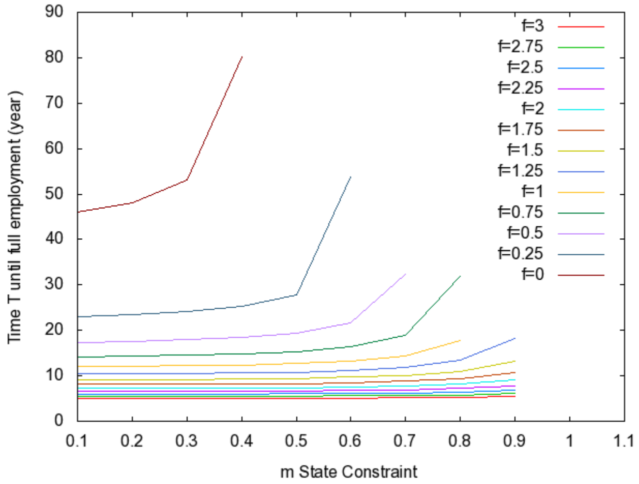

Fig. 4. The relation between foreign aid f, full employment time T, and minimum per capita consumption m with disturbances.

6.2 *Calculating the Optimal Control for the Foreign Aid Model*

In the following, the full employment time is taken from the above results depending on the system being considered and then we simulate the corresponding system using stored grid values of the control. At each system we compare between the results obtained without considering a disturbance and the results when a disturbance is considered.

We start with the two dimensional foreign aid system (10) with foreign aid factor f equals 0.5. Without considering any disturbances, the full employment time T is 13 years (from Fig. 1). Subsequently, when adding a disturbance the full employment time T is 17 years (from Fig. 2). We calculate the value function of the transformed system (16) with a grid of size 700×700, the time step equals 5×10^{-4}, and the number of parallelization threads is 30. The calculated optimal controls u in both cases are represented in Figure 5. Without disturbances, the resulting investment program is:
Between years 0 *and* 11, investment goes to the capital sector only.
Between years 11 *and* 13, investment goes to the consumption sector only.

After considering disturbances, the resulting investment program is:
Between years 0 *and* 15, investment goes to the capital sector only.
Between years 15 *and* 17, investment goes to the consumption sector only.

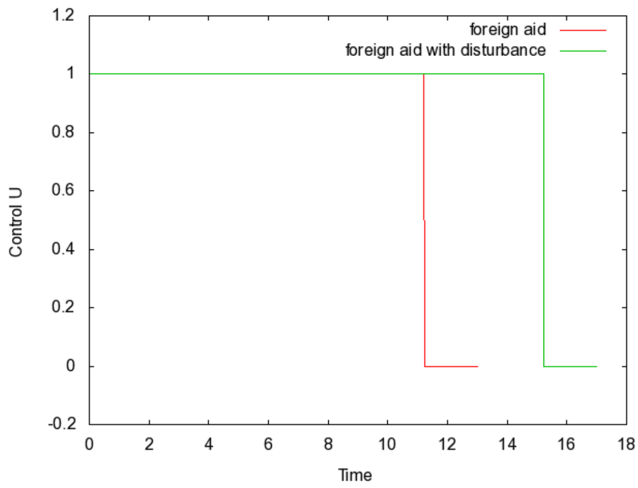

Fig. 5. The optimal control *(u)* of the foreign aid system with and without disturbance.

When state constraint (11) is added to the foreign aid system (10) with foreign aid factor f equals 0.5 and minimum per capita consumption m equals 0.4, the full employment time T equals 14.6 years (from Fig. 3) without considering any disturbances and equals 18 years when a disturbance is considered (from Fig. 4). The calculated optimal controls u in both cases are represented in Figure 6. This bang-bang control is non-applicable in the economical control, and an average of the optimal control values can be calculated, for example:

$$\tilde{u}(t_n) = \frac{1}{20}\sum_{k=n-10}^{n+10} u(t_k),\quad 10 \le n \le M-10. \qquad (29)$$

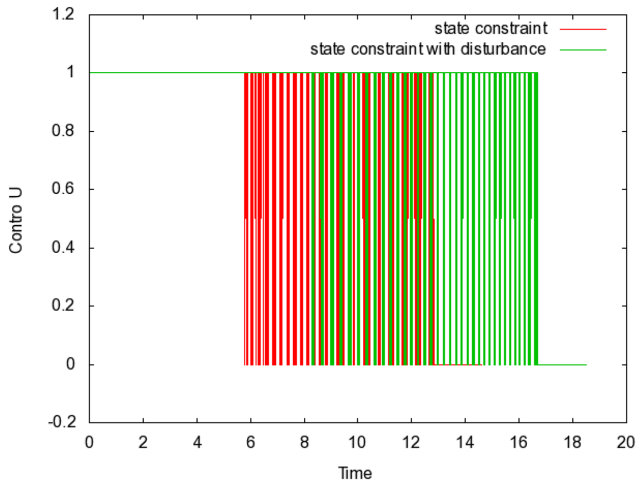

Fig. 6. The optimal control *(u)* of the foreign aid system with state constraint and with and without disturbance.

6.3 *Calculating the Optimal Control for the Extended Export Model*

We calculate the full employment time T for the three dimensional extended export model (13) using transformed system (19) as follows:

$$\hat{T} = \min\{\hat{\theta} : \hat{V}_{\hat{\theta}}(0,\hat{x}_0,\hat{y}_0,\hat{z}_0) = 0\}, \qquad (30)$$

where

$$\hat{V}_{\hat{\theta}} = \min_{u_x(\cdot),u_y(\cdot),u_z(\cdot)} \min_{t\in[0,\hat{\theta}]} \left(1 - \left(\lambda\, \hat{x}(t) + \delta\, \hat{z}(t) + \hat{y}(t)\right)\right),$$

$$\text{and } T = 10\,\hat{T}\,.$$

The value function is calculated with a grid of size 200×200 ×200, the time step equals 5×10^{-4}, the number of parallelization threads is 30, and we chose foreign aid factor f to be equal to 1.0. The resulting full employment time T is 1.7 years. One can see that adding just one additional sector that generates more money for investment decreases the time required for full employment significantly.

By applying the numerical method, the resulting optimal controls for the export, capital, and consumption sectors are presented in Figure 7, Figure 8, and Figure 9 respectively. The resulting investment program is:
Between years 0 *and* 1.3, investment goes to the export sector only.
Between years 1.3 *and* 1.7, investment goes to the consumption sector only.

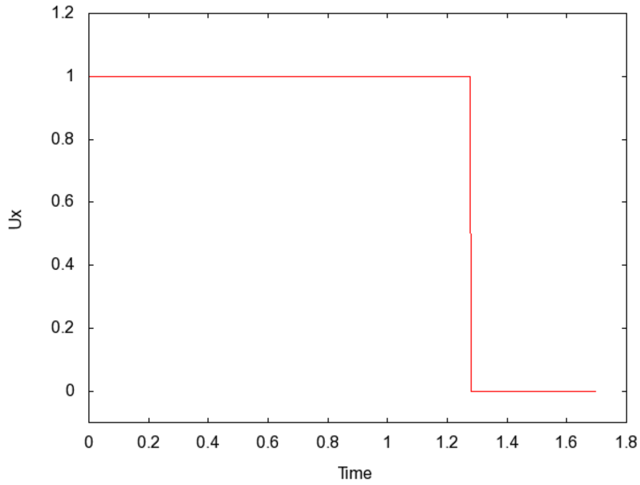

Fig. 7. The optimal fraction of investment *(U_x)* into the Export sector of the extended Export model.

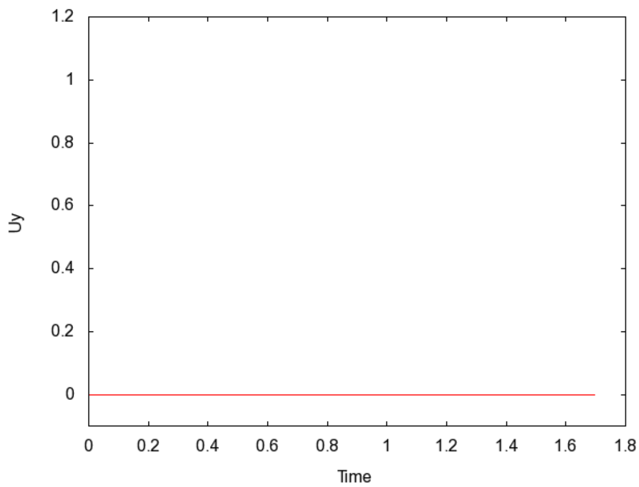

Fig. 8. The optimal fraction of investment *(U_y)* into the Capital sector of the extended export model.

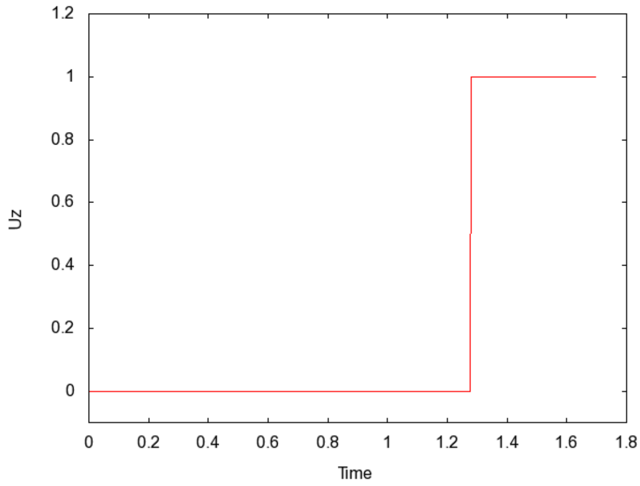

Fig. 9. The optimal fraction of investment *(U_z)* into the Consumption sector of the extended export model.

6. CONCLUSION

The aim of this work is to apply dynamic programming methods to an enlarged version of Stoleru's economic model. By following this approach, we are able to consider more complicated economical models including the presence of disturbances and complicated state constraints. The obtained results are qualitatively consistent with the results obtained in Yassouridis (2010) by using Pontryagin's maximum principle. Additionally, the applied method allows us to consider non linear problems which are important if we want to present the economy with all its influences and unpredictable behaviour fairly.

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